N4 Mathematics

Gateways to Engineering Studies - John Dillon & Chris Brink

HYBRID LEARNING SOLUTIONS

Gateways to Engineering Studies

Mathematics

N4

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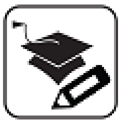
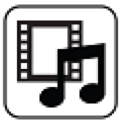
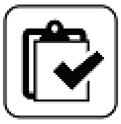
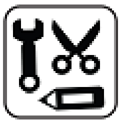
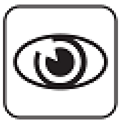
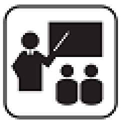
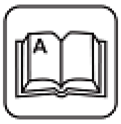
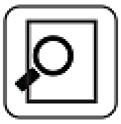
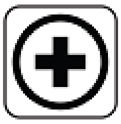
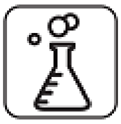
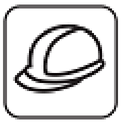
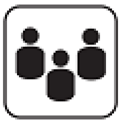
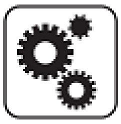
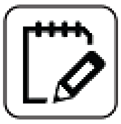
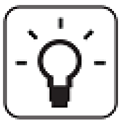
	Assessment / Activity		Multimedia
	Checklist		Practical
	Demonstration/ observation		Presentation/ Lecture
	Did you know?		Read
	Example		Safety
	Experiment		Site visit
	Group work/ discussions, role-play, etc.		Take note of
	In the workplace		Theoretical – questions, reports, case studies, etc.
	Keywords		Think about it

Module 1

Equations, Manipulation and Word Problems

Learning Outcomes
On the completion of this module the student must be able to:

- Describe algebraic terms commonly used
- Solve equations and cubic equations using factorization
- Solve exponential equations
- Solve exponential equations using logarithms
- Translate and solve word problems
- Solve simultaneous equations

1.1 Introduction

This module discusses commonly used algebraic terms. It also demonstrates how to solve various equations using different methods and the solving of word problems.

1.2 The language of algebra

For revision purposes, go over the basic mathematical concepts and operations that lead up to factorising complex expressions.

1.2.1 Terms, factors and expressions

Definition: Term
A term is either a single number or a variable, or numbers and variables multiplied together. Terms are added or subtracted from one another.

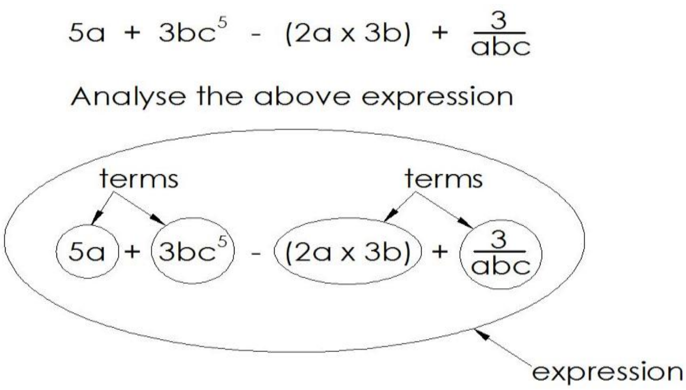


Figure 1.1 An expression with four terms

In **Figure 1.2**, the 3 is a factor and so is the b^5 a factor.

Definition: Factor
A factor is a group of numbers, coefficients or variables that are divided or multiplied to each other.

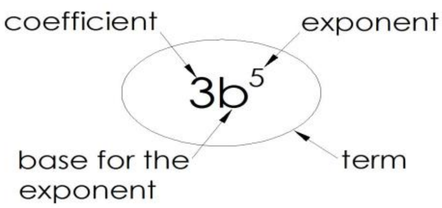


Figure 1.2

Definition: Expression
An expression is a group of terms (the terms are separated by + or – signs).

1.2.2 Equations

Definition: Equation
An equation says that two things are equal. It will have an equalssign "=".

Worked Example 1.1

Make E the subject of the following equation. Do not simplify:

$$5E + 3F^2 - (2 \times F) = 2$$

Solution:

$$5E + 3F^2 - (2 \times F) = 2$$

$$SE = 2 + (2 \times F) - 3F^2$$

$$E = \frac{2 + (2 \times F) - 3F^2}{5}$$

1.3 Exponents

Definition:
The exponent (such as the 2 in x^2) says how many times to use the value in a multiplication. Other words for exponent is index or power.

The power form can often be used to help solve arithmetical problems quickly.

1.3.1 Laws of exponents

Table 1.1 shows the laws that apply to exponents:

No.	Expression	Applied law
1	$a^P \times a^L$	a^{P+L}
2	$(a^P)^L$	$a^{P\,L}$
3	$\sqrt[P]{a}$	$a^{\frac{1}{P}}$
4	$(Anything)^0$	1
5	$a^P \div a^L$	a^{P-L}
6	a^{-P}	$\frac{1}{a^P}$
7	$\sqrt[L]{a^P}$	$a^{\frac{P}{L}}$

Table 1.1 Laws of exponents

Worked Example 1.2

The following expressions are simplified using the laws of exponents:

$$(3\,a^3\,b^2\,c)^2 = 3^{1+2}\,a^{3+2}\,b^{2+2}\,c^{1+2} = 9\,a^5\,b^4\,c^2$$

$(a^4\,b^{12})^{\frac{1}{2}} \;=\; a^{\frac{4}{2}}\,b^{\frac{12}{2}} \;=\; a^2\,b^6$

$\sqrt{36\,a^2\,b^6} \;=\; 36^{\frac{1}{2}}\;a^{\frac{2}{2}}\;b^{\frac{6}{2}} \;=\; 6\,a\,b^3$

$\sqrt[3]{27\,a^0\,b^6} \;=\; \sqrt[3]{27\;\times 1\;\times\;b^6} \;=\; 27^{\frac{1}{3}}\;\;b^{\frac{6}{3}} \;=\; 3\,b^2$

1.4 Factorise the sum or difference of two cubes

Definition: Factorise
Factorising is the process of finding the factors. It is like "splitting" an expression into a multiplication of simpler expressions.

1.4.1 Removing the highest common factor method

Whenever we factorise, use this method before using any other factorisation method.

Worked Example 1.3

Factorise the expression $6ax + 18ay - 12bx + 6by$

Solution:

$$6(ax + 3ay - 2bx + by)$$

1.4.2 Grouping

In the expression $ax + ay + bx + by$ there is no common factor, but there is a common factor between the first two and the last two terms.

Worked Example 1.4

Factorise the expression: $ax + ay + bx + by$

Solution:

$$a(x + y) + b(x + y)$$

The common factor is $(x + y)$

$$(x + y)(a + b)$$

1.4.3 Difference between two squares method

If the sum of two digits is multiplied by their difference, then the difference between two squares is obtained.

For example:

$(x+y)(x-y)$

$= x^2 - xy + xy - y^2$

$= x^2 - y^2$

Worked Example 1.5

$$8 - 32a^2$$

Solution:

Take out the common factor first:

$$8(1 - 4a^2)$$

Then:

$$8(1+2a)(1-2a)$$

1.4.4 Difference or sum of two cubes method

Worked Example 1.6

Factorise the following expression:

$$a^3 - b^3$$

Solution:

Write down two brackets:

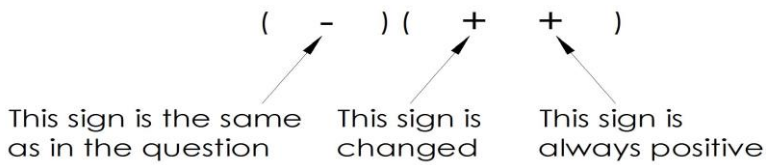


Figure 1.3

In the first bracket, write down the cube root of a^3 and b^3.

$$(a - b)\ (\ \ +\ \ \ +\ \ )$$

Then in the second bracket write down the first term in the first bracket squared. Do the same with the second term.

$$(a - b)\ (a^2 + \quad + b^2)$$

Then, for the centre term in the second bracket, write down the product of the two terms in the first bracket.

$$(a - b)\ (a^2 + ab + b^2)$$

Check this answer by multiplication. You will arrive at: $a^3 - b^3$

1.4.5 Factorising the quadratic trinomial

The first thing to look for is a common factor in the three terms. First take out the common factor if there is one and then proceed as follows:

Worked Example 1.7

Factorise the expression $x^2 - 2x - 3$

Solution:

Write down two brackets:

$$(\quad\quad)(\quad\quad)$$

Enter the two factors of the first term:

$$(x \quad\quad)(x \quad\quad)$$

Enter the factors of the last term:

$$(x \quad\quad 3)(x \quad\quad 1)$$

Check if this is correct by adding or subtracting the product of the outer two factors with the inner two factors:

$$x \times 1 \quad add \quad 3 \times x \quad this\ gives\ 4x\ \ which\ does\ not\ equal\ the\ middle\ term$$

$$x \times 1 \quad subtract \quad 3 \times x \quad this\ gives\ -2x\ \ which\ does\ equal\ the\ middle\ term$$

Now make sure the signs are correct so that when the brackets are multiplied, they produce the quadratic trinomial which was the question.

$$(x - 3)(x + 1)$$

Did you know?

Figure 1.4

Albert Einstein (1879-1955)
Nationality: German, American
Famous For: *E=m*c*^{2}

Albert Einstein excelled in mathematics early in his childhood. He liked to study math on his own. He was once quoted as saying, "I never failed in mathematics…before I was fifteen I had mastered differential integral calculus."

1.5 Solving equations using logarithms

The **Table 1.2** below shows the laws that apply to Logarithms:

No.	Logarithmic law	Logarithmic expression
1	$\log a + \log b$	$\log ab$
2	$\log a - \log b$	$\log \frac{a}{b}$
3	$\mathrm{n} \log a$	$\log a^n$
4	$\frac{\log_c a}{\log_c b}$	$\log_b a$

Table 1.2 Laws of Logarithms

The **Table 1.3** Logarithms that are commonly used:

No.	Logarithmic expression	Value
1	$\log_{10} 10$	1
2	$\log_{10} 100$	2
3	$\log_{10} 7.2125$	$\log 7.2125$
4	$\log_b b$	1
5	$\log_{10} 0.1$	-1

Table 1.3 Commonly used logarithms

Did you know?

Figure 1.5

John Napier (1550-1617)
Nationality: Scottish
Famous For: *Inventing "logarithms"*

John Napier is responsible for manufacturing logarithms. It was also he who applied the everyday use of the decimal point in mathematics and arithmetic. Napier's bones was an abacus created by John. The device was used mainly for multiplication problems.

Worked Example 1.8

Find the antilogarithms of the following:

1. $\ln 17 = 2.83$

2. $\log_3 27$

Solution:

1. $\log_e 17 = 2.83$

$$17 = e^{2.83}$$

2. $\log_3 81 = 4$

$$81 = 3^4$$

Worked Example 1.9

Make Q the subject of the following:

$$110 = ab^Q$$

Solution:

$$110 = ab^Q$$

$$ab^Q = 110$$

$$b^Q = \frac{100}{a}$$

$$\log b^Q = \log \frac{100}{a}$$

$$\text{Q} \log b = \log 100 - \log a$$

$$\text{Q} = \frac{\log 100 - \log a}{\log b}$$

Worked Example 1.10

Make y the subject of $\ell n\sqrt{1-y} - \ell n\sqrt{1+y} = x$

Solution:

$$\ell n\sqrt{1-y} - \ell n\sqrt{1+y} = x$$

$$\therefore \ell n\frac{\sqrt{1-y}}{\sqrt{1+y}} = x$$

$$\therefore \sqrt{\frac{1-y}{1+y}} = e^x$$

$$\therefore \frac{1-y}{1+y} = [e^x]^2$$

$$\therefore 1-y = e^{2x}(1+y)$$

$$= e^{2x} + ye^{2x}$$

$$\therefore 1-e^{2x} = ye^{2x} + y$$

$$= y[e^{2x}+1]$$

$$\therefore y = \frac{1-e^{2x}}{e^{2x}+1}$$

1.5.1 Using logarithms to solve equations

Worked Example 1.11

Find the value of a:

$$\log 3 \log a = \log 9$$

$$\log a = \frac{\log 9}{\log 3}$$

$$\log a = \frac{\log 3^2}{\log 3}$$

$$\log a = \frac{2\log 3}{\log 3}$$

$$\log a = 2$$

Write $\log a = 2$ in exponential form $10^2 = a$

$$a = 100$$

Worked Example 1.12

The equation $x^{2a} - 3x^a + 2 = 0$ can be solved by substituting

$$k = x^a$$

The equation becomes $k^2 - 3k + 2 = 0$ which can be solved:

$$k^2 - 3k + 2 = 0$$

$$(\text{k} - 1)(k - 2) = 0$$

$$k = 1 \quad or \quad k = 2$$

$$10^k = 1 \quad or \quad 10^k = 2$$

Take logs $\quad \log 10^k = \log 1 \quad or \quad \log 10^k = \log 2$

Log law $\quad k \log 10 = \log 1 \quad or \quad k \log 10 = \log 2$

$\log 10 = 1$ Therefore $\quad k = 0 \quad or \quad k = \log 2$

1.6 Solving equations

1.6.1 Solving exponential equations

Exponential equations can be solved by using the laws discussed earlier.

Worked Example 1.13

Find the value of x:

$$0.7^x = 1$$

Solution:

$$(0.7)^x = (0.7)^0 \quad \ldots \quad 10^0 = 1$$

$x = 0 \quad$ *Same* bases are dropped

Worked Example 1.14

Solve for a and b: $4^{a+b} = 2^{b+4}$ $\quad and \quad b = \frac{a}{2}$

$$b = \frac{a}{2}$$

$$a = 2b \quad \ldots \quad (1)$$

$$4^{a+b} = 2^{b+4}$$

$$2^{2a+2b} = 2^{b+4}$$

$$2a + b = 4 \quad \ldots \quad (2)$$

$$\text{Substitute (1) into (2)} \quad 2(2b) + b = 4 \quad \ldots \quad (2)$$

$$5b = 4$$

$$b = \frac{4}{5}$$

$$a = 2\left(\frac{4}{5}\right) \quad \ldots \quad (1)$$

$$a = \frac{8}{5}$$

1.6.2 Solving simultaneous equations with three unknowns

Three equations can be used to solve three unknowns. One of the unknowns needs to be eliminated from all three equations. The following worked example will show how this is done.

Worked Example 1.15

Solve the three unknowns in the three equations below:

$$4c - 2b = 14 - 3a \quad \ldots \quad (1)$$

$$a - b - c - 3 = 0 \quad \ldots \quad (2)$$

$$5 - 2a = -3b - 3c \quad \ldots \quad (3)$$

Solution:

Rewrite in order:

$$3a - 2b + 4c = 14 \quad \ldots \quad (1)$$

$$a - b - c = 3 \quad \dots \quad (2)$$

$$-2a + 3b + 3c = -5 \quad \dots \quad (3)$$

Eliminate a by subtracting (2) from (1):

$$3a - 2b + 4c = 14 \quad \dots \quad (1)$$

$$\underline{3a - 3b - 3c = 9} \quad \dots \quad (2) \quad \times 3$$

$$b + 7c = 5 \quad \dots \quad (4)$$

Eliminate a by subtracting (3) from (2):

$$-2a + 2b + 2c = -6 \quad \dots \quad (2) \quad \times -2$$

$$\underline{-2a + 3b + 3c = -5} \quad \dots \quad (3)$$

$$-b - c = -1 \quad \dots \quad (5)$$

Eliminate b by subtracting (5) from (4):

$$b + 7c = 5 \quad \dots \quad (4)$$

$$\underline{b + c = 1} \quad \dots \quad (5) \quad \times -1$$

$$6c = 4$$

$$\boldsymbol{c} = \frac{4}{6} = \frac{2}{3}$$

Substitute into (5):

$$b + \frac{2}{3} = 1 \quad \dots \quad (5)$$

$$\boldsymbol{b} = \frac{1}{3}$$

Substitute into (2):

$$a - \frac{1}{3} - \frac{2}{3} = 3 \quad \dots \quad (2)$$

$$\boldsymbol{a = 4}$$

1.7 Translate Word problems

Often, a simple problem becomes difficult to solve because the mathematical terms used are not fully understood. The meaning of some of the terms used in word problems are given below.

1.7.1 Converting the written word into mathematical language

No.	Written word	Translation
1	The sum of two numbers	$a + b$
2	The difference of two numbers	$a - b$
3	The product of two numbers	$a \times b$
4	The quotient of two numbers	$\frac{a}{b}$
5	The sum of the square of two numbers	$a^2 + b^2$
6	The sum of two numbers squared	$(a + b)^2$
7	The sum of three consecutive numbers	$(a) + (a + 1) + (a + 2)$
8	A number added to its reciprocal	$a + \frac{1}{a}$

Table 1.4 Translating the written word into mathematical expressions

1.7.2 Solving word problems with two unknowns

The statement that reads:

The sum of the squares of two consecutive numbers is 100.
is written as follows:

Let the first number equal a:

$$(a)^2 + (a + 1)^2 = 100$$

A number decreased by 2 equals 14 times its reciprocal.
Is written as follows:

Let the number equal b:

$$b - 2 = 14 \times \frac{1}{b}$$

Worked Example 1.16

The difference between two numbers is 7. The sum of their squares is 29. Find the numbers.

Solution:

$x - y = 7$ ………………….. (1)

$x^2 + y^2 = 29$ ………………… (2)

ln(1) $x = 7 + y$

ln(2) $(7 + y)^2 + y^2 = 29$

$y^2 + 7y + 10 = 0$

$(y + 5)(y + 2) = 0$

$$y = -5 \ and \ y = -2$$

$$\ln(1) \ x + 5 = 7 \ and \ x + 2 = 7$$

$$x = 2 \ and \ x = 5$$

$$\therefore x = 5 \ and \ y = -2$$

Or

$$x = 2 \ and \ y = -5$$

Worked Example 1.17

The length of a rectangle is equal to three times the breadth. The area of the rectangle is 48 m^2. Calculate the length and the breadth of the rectangle.

Solution:

$$\ell = 3b \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$\ell \times b = 48 \ldots\ldots\ldots\ldots. (2)$$

$$3b.b = 48$$

$$b^2 = 16$$

$$b = 4m$$

$$\ell = 3 \times 4$$

$$= 12 \ m$$

As an example of solving word problems, take population Problems where logarithms to the base **e** are used:

To solve an exponential or logarithmic word problem, convert the written word to an equation and solve the equation.

In this example, we will review population problems. We will also discuss why the base of **e** is used so often with population problems.

Worked Example 1.18

Suppose that you are observing the behaviour of cell duplication in a lab. In one experiment, you started with one cell and the cells doubled every minute.

Write an equation with base 2 to determine the number (population) of cells after one hour.

Solution:
First record your observations by making a table with two columns:
One column for the time and one column for the number of cells.

The number of cells (size of population) depends on the time. If you were to graph your findings, the points would be formed by (specific time, number of cells at the specific time).

For example, at t = 0, there is 1 cell, and the corresponding point is (0, 1).
At t = 1, there are 2 cells, and the corresponding point is (1, 2).
At t = 2, there are 4 cells, and the corresponding point is (2, 4).
At t = 3, there are 8 cells, and the corresponding point is (3, 8).

It appears that the relationship between the two parts of the point is exponential. At time 0, the number of cells is 1 or $2^0 = 1$.
After 1 minute, when t = 1, there are two cells or $2^1 = 2$.
After 2 minutes, when t = 2, there are 4 cells or $2^2 = 4$.

Therefore, one formula to estimate the number of cells (size of population) after t minutes is the equation:

$$f_t = 2^t$$

Worked Example 1.19

Using the equation developed in **Worked example 1.18** do the following:
Determine the number of cells after one hour:

Solution:

One hour to minutes = 60 min

Substitute 60 for t in the equation. $f(t) = 2^t$:

$$f_{60} = 2^{60} = 1.15 \times 10^{18}$$

Worked Example 1.20

Using the equation developed in **Worked example 1.18** do the following:

Determine how long it would take the population (number of cells) to reach 100,000 cells.

Solution:
In this example, you know the number of cells at the beginning of the experiment (1) and at the end of the experiment (100,000), but you do not know the time.

Substitute 100,000 for f_t in the equation $f_t = 2^t$

$100,000 = 2^t$

Take the natural logarithm of both sides:

$$\ln(100{,}000) = \ln(2^t)$$

Simplify the right side of the equation using the third rule of logarithms:

$$\ln(100{,}000) = t\ln(2)$$

Divide both sides by $\ln(2)$:

$$\frac{\ln(100{,}000)}{\ln(2)} = t$$

$$t = 16.60964\ min$$

It would take 16.6 minutes, rounded, for the population (number of cells) to reach 100,000.

Activity 1.1

1. $a^3 + b^3$
2. $3a^3 + 3b^3$
3. $(a+b)^3 - 4^3$
4. $(x+2y)^3 + (x+y)^3$
5. $a^3 - 27b^3$

6. $8x^4 - 27xy^3$

7. $27x^3 - (2x + y)^3$

8. $8 + (2 + y)^3$

Answers:

1. $(a + b)(a^2 - ab + b^2)$

2. $3(a + b)(a^2 - ab + b^2)$

3. $\big((a + b) - 4\big)((a + b)^2 + 4(a + b) + 16)$

4. $(2x + 3y)(3xy + 3y^2 + x^2)$

5. $(a - 3b)(a^2 + 3ab + 9b^2)$

6. $x(2x - 3y)(4x^2 + 6xy + 9y^2)$

7. $\big((x + 2y) - 3y\big)((x + 2y)^2 + 3y(x + 2y) + 9y^2)$

8. $(4 + y)(y^2 - 2y + 4)$

Activity 1.2

Make the quantity shown in brackets the subject of the following formula:

1. $b = \log_4 a$ ... (a)

2. $k = me^{-Pt}$ ... (t)

3. $I = \frac{V}{\sqrt{R^2 + X^2}}$ ... (X)

4. $\frac{T_1}{T_2} = e^{\mu\theta}$ ... (θ)

5. $S = Ut + \frac{1}{2}at^2$ ... (t)

Answers:

1. 5^b

2. $t = \frac{\log k - \log M}{-P \log e}$

3. $X = \frac{\pm\sqrt{V^2 - I^2R^2}}{I}$

4. $\theta = \frac{\log T_1 - \log T_2}{U \log e}$

5. $t = \frac{-U \pm \sqrt{U^2 + aS}}{a}$

Activity 1.3

Solve for a in the following:

1. $2^{(2a+1)} + 2^{(2a+1)} = 8$

2. $3^{a^2+5a+6} = 1$

3. $\left(\frac{1}{3}\right)^a = 81$

4. $2^a = 8$

5. $5^{3a} = \frac{1}{5}$

Answers:

1. $\frac{1}{2}$

2. $a = -2 \quad or \quad a = -3$

3. $a = -4$

4. $a = 3$

5. $a = \frac{-1}{3}$

Activity 1.4

Solve for a in the following:

1. $2^{2a} - 6.2^{a} + 8 = 0$

2. $10^{2a} + 10^{a} - 10^{a+2} - 10^{2} = 0$

3. $5^{2a} - 4.5^{a+1} - 5^{3} = 0$

4. $e^{2a} - (e+1)e^{a+1} + e^{3} = 0$

Answers:

1. 1 : 2

2. 2

3. 2

4. 1 : 2

Activity 1.5

Change the subject of the following formulae to the symbol as shown in brackets:

1. $p = q(1 + xt) \ldots\ldots\ldots.(x)$

2. $\frac{a^2}{1-ac} = \frac{b^2}{a-bc} \ldots\ldots\ldots(c)$

3. $d = D\sqrt{\frac{p+1}{p-1}\ldots\ldots\ldots..(f)}$ fdf

4. $c \times \sqrt[n]{\frac{y}{x}} = \frac{a}{2b} \ldots\ldots\ldots(x)$

Answers:

1. $x = \frac{P-q}{qt}$

2. $c = \frac{b+a}{a\,b}$

3. $f = \frac{P(d^2 - D^2)}{D^2 - d^2}$

4. $x = y\left(\frac{2bc}{a}\right)^n$

Activity 1.6

1. $x^{2/3} = 64$

2. $4^{-x} - 16 = 0$

3. $2^{x+2} + 2^{x-1} = 6^{x-1}$

4. $2^x + 4 = 0$

5. $2 \times 2^{2x} - 6 \times 2^{2x} + 4 = 0$

Answers:

1. 512

2. -2

3. 3

4. No solution

5. 0 : 1

Activity 1.7

1. $3^{a+3} = 9^b$ *and* $2^{a-3} = 16^b$

2. $5^{a+1} = 25^b$ *and* $2^a = 8^{b+2}$

3. $27 = 9^{a-1} + 3^b$ *and* $5^a = 25^b$

Answers:

1. *and* $b = -3$

2. $a = -15 \quad and \quad b = -7$

3. $a = \frac{10}{3} \quad and \quad b = \frac{5}{3}$

Activity 1.8

Solve for the unknowns in the following equations:

1. Given:
   $2a + 3b = 14$
   $3a - 2b = -5$
   Solve for a and b:

2. Given:
   $3y + 2x = z + 1$
   $3x + 2z = 8 - 5y$
   $3z - 1 = x - 2y$
   Solve for y:

3. Given:
   $a + 4b + 8c = 0$
   $2a - 5b = -2 + 6c$
   $b - 4c = -3$
   Solve for c:

Answers:

1. $y = 1$

2. $a = 1 \quad and \quad b = 4$

3. $c = 0.554$

Activity 1.9

1. The hypotenuse of a right –angled triangle is 10 mm longer than the longest of the two other sides. Calculate the lengths of the sides if the shortest side is 50 mm long.
2. Determine the length and breadth of a rectangle if the length of a rectangle is 3 m more than its breadth and the area of a rectangle is 54 m^2.
3. The sum of two numbers is 10. The difference of their squares is 50. Calculate TWO numbers.

4. $\frac{1}{3}$ of the difference between two numbers is 10. 4 times their sum is 200. Determine the two numbers.

Answers:

1. 120 : 130

2. 6 : 9

3. 2.5 : 7.5

4. 40 : 10

Activity 1.10

1. Solve for x if:

   $5(3^{2x+2}) = 31$

2. Given: $i = I(e_1^2 + 1)$

   Make g the subject of the formulae

3. Solve for x if:

   $e^{x-1} = 7(2^x)$

4. Given

   $10e^{3t} = 51$

   Make t the subject of the formula.

5. Make P_2 the subject of the formula if:

   $H = 100\log\frac{P_1}{P_2}$

6. Solve for x if:

   $5^{x+1} = 2^{x-2}$

7. Solve for x if:

   $3.3^x = 28$

8. Make y the subject of the formula if:

   $q = 3p\left[2 - \log\left(\frac{y}{2}\right)\right]$

9. $t = \frac{w}{g}\log\left(1 + \frac{y}{z}\right)$

   Make y the subject of the formula.

10. Solve for x if:

    $2^{5x-2} = 5^{3x+1}$

11. Given:

    $T = 15 + 350e^{kt}$

    Make k the subject of the formula.

12. Solve for x if:

    $\left(\frac{1}{3}\right)^{x-2} = 6^{x+2}$

Answers:

1. $x = -0.169$

2. $g = \frac{V}{\ln\left(\frac{I}{i} - 1\right)}$

3. $x = 9.596$

4. $t = \frac{\ln(0.51)}{3}$

5. $P_2 = \frac{P_1}{\frac{10\,H}{100}}$

6. $x = -3.226$

7. $x = 2.036$

8. $y = 2\left(10^2\ \frac{q}{3p}\right)$

9. $y = Z\left(10\frac{tg}{w} - 1\right)$

10. $x = -2.1985$

11. $k = \frac{\ln\left(\frac{T-15}{350}\right)}{t}$

12. $x = -0.481$

Self-Check

I am able to:	**Yes**	**No**
• Describe algebraic terms commonly used		
• Solve equations and cubic equations using factorization		
• Solve exponential equations		
• Solve exponential equations using logarithms		
• Translate and solve word problems		
• Solve simultaneous equations		

If you have answered ‘no’ to any of the outcomes listed above, then speak to your facilitator for guidance and further development.

Module 2

Determinants

Learning Outcomes
On the completion of this module the student must be able to:

- Solve simultaneous equations with two or three unknowns
- Calculate determinants and apply Cramer's rule
- Calculate the minor of a third order determinant
- Define the co-factor of the minor

2.1 Introduction

In linear algebra, the determinant is a useful value that can be computed from the elements of a square matrix. The determinant of a matrix A is denoted det(A), det A, or |A|.

2.1.1 Determinant notation

$$\text{Det A} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

The matrix above shows how a determinant is written. This is a determinant of order two… Commonly called a second order determinant.

It has two rows, two columns and four elements.

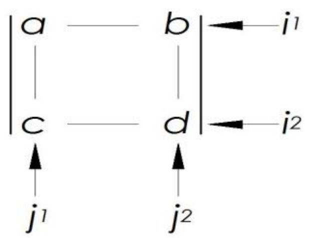


Figure 2.1

The four elements are *a,b,c* and *d*. An example of how they are written is shown below.

Note:
The sub-script of each element shows the row first and then the column.

Element *a* - row 1 - column 1 - is written a_{ij} or it can be written a_{11}

Element *b* - row 1 - column 2 - is written a_{12}

Element *c* - row 2 - column 1 - is written a_{21}

Element *c* - row 2 - column 2 - is written a_{22}

In the case of a second order matrix, the specific formula for the determinant is simply the upper left element times the lower right element, minus the product of the other two elements.

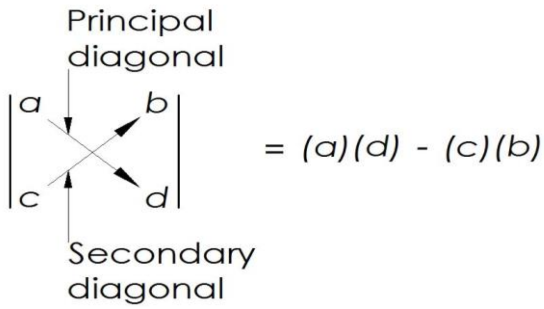


Figure 2.2

Worked Example 2.1

Evaluate the following determinant:

$$\begin{vmatrix} -3 & -2 \\ -1 & 5 \end{vmatrix}$$

Solution:

$$D = (-3)(5) - (-1)(-2)$$

$$D = -15 - 2$$

$$D = -17$$

2.1.2 Determinant of order 3

Similarly, suppose we have a 3 × 3 matrix A, also called a determinant of order 3 and we want the specific formula for its determinant |A|:

Use the Diagonal expansion method to determine the value as shown in the **Worked Example 2.2**.

Worked Example 2.2

Evaluate the determinant of order 3:

$$A = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & -8 & 9 \end{vmatrix}$$

Figure 2.3

Solution:

Use the diagonal expansion method to determine the value:

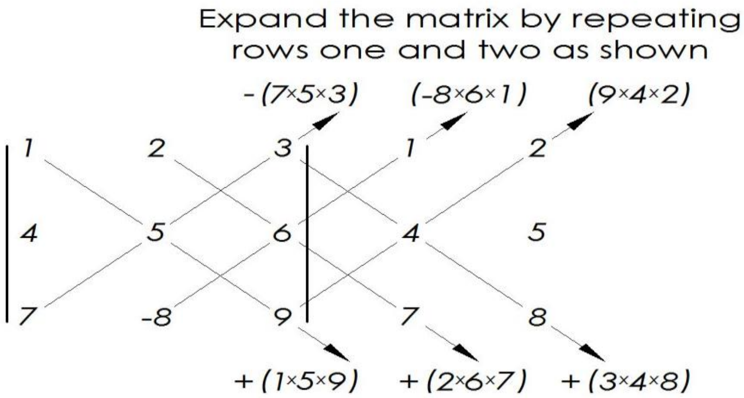


Figure 2.4

$$A = (1 \times 5 \times 9) + (2 \times 6 \times 7) + (3 \times 4 \times 8) - (7 \times 5 \times 3) - (-8 \times 6 \times 1) - (9 \times 4 \times 2)$$

$$A = (45) + (84) + (96) - (105) - (-48) - (72)$$

$$A = 96$$

2.2 Minors and co-factors

2.2.1 The minor of an element

When working with a determinant of order 3. The minor of an element can be found by deleting the row and column of the element in the 3x3 matrix and then find the determinant of the remaining second order determinant.

Worked Example 2.3

In the determinant of order 3 below, find the minor of element 1:

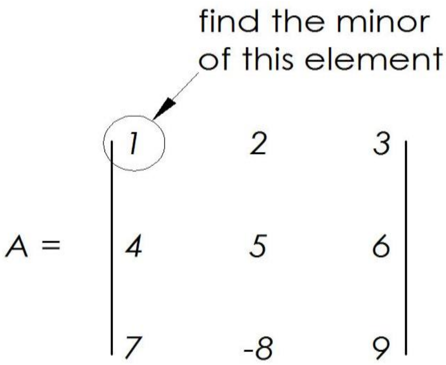


Figure 2.5

Solution:

Delete the row and column in which the element is found:

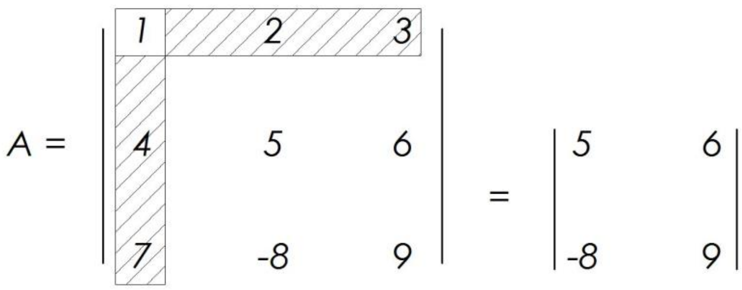


Figure 2.6

Find the value of the 2x2 determinant:

$$\Delta = (5 \times 9) - (-8 \times 6)$$

$$\Delta = (5 \times 9) - (-8 \times 6)$$

$$\Delta = 93$$

2.2.2 The co-factor of an element

The co-factor is a signed minor. It is written as a formula:

$$A_e = (-1)^{(i+j)} \times M_e$$

Where:
A is the co-factor
M is the minor
i is the row number
j is the column number

Worked Example 2.4

In the determinant of order 3 below, find the minor of element 1 and the co-factor:

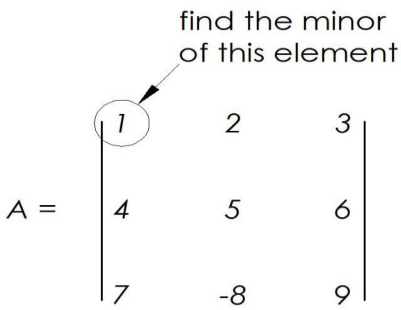


Figure 2.7

Solution:

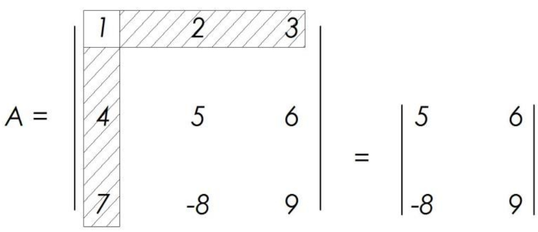


Figure 2.8

$$The\ minor\ \ M = 93$$

i is the row number 1
j is the column number 1

$$A_e = (-1)^{(i+j)} \times M_e$$

$$A_1 = (-1)^{(1+1)} \times M_1$$

$$A_1 = 1 \times 93$$

$$A_1 = 93$$

Worked Example 2.5

In the determinant of order 3 below, find the minor of element 5 and the co-factor:

$$A = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & -8 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ 7 & 9 \end{vmatrix}$$

Figure 2.9

$$\text{The minor } M_5 = (1 \times 9) - (7 \times 3)$$

$$M_5 = (9) - (21) = -12$$

i is the row number 2

j is the column number 2

$$A_e = (-1)^{(i+j)} \times M_e$$

$$A_5 = (-1)^{(2+2)} \times (-12)$$

$$A_5 = 1 \times (-12)$$

$$A_5 = -12$$

2.3 Calculating third order determinants using the co-factor expansion method

Below is a 3x3 matrix of negative and positive coefficients:

$$A = \begin{vmatrix} -1 & -2 & -3 \\ -4 & -5 & -6 \\ 7 & 8 & 9 \end{vmatrix}$$

Figure 2.10

The sign that forms part of the co-factor is found by looking at the row (i) and the column (j). It can also be found by formula:

$$(-1)^{(i+j)}$$

In the 3x3 matrix below, notice how the index of each coefficient affects the sign used with the co-factor.

$$A = \begin{vmatrix} (-1)^{1+1} & (-2)^{1+2} & (-3)^{1+3} \\ (-4)^{2+1} & (-5)^{2+2} & (-6)^{2+3} \\ (7)^{3+1} & (8)^{3+2} & (9)^{3+3} \end{vmatrix} = \begin{vmatrix} (1) & (-2) & (3) \\ (-4) & (5) & (-6) \\ (7) & (8) & (9) \end{vmatrix}$$

Figure 2.11

The third order determinant is calculated by expanding along the first row. That is -1, -2 and -3.

$$A = \begin{vmatrix} -1 & -2 & -3 \\ -4 & -5 & -6 \\ 7 & 8 & 9 \end{vmatrix}$$

Figure 2.12

Doing this means finding the co-factors of the elements in the first row:

$$A = (-1)^{1+1} \times a_{-1} \begin{vmatrix} -5 & -6 \\ 8 & 9 \end{vmatrix} + (-1)^{1+2} \times a_{-2} \begin{vmatrix} -4 & -6 \\ 7 & 9 \end{vmatrix} + (-1)^{1+3} \times a_{-3} \begin{vmatrix} -4 & -5 \\ 7 & 8 \end{vmatrix}$$

Note:
When the first element in the first row is chosen, delete the row and column in which the element is found and calculate the second order determinant.

$$A = (1)(-1)[(-45) - (-48)] + (-1)(-2)[(-36) - (-42)] + (1)(-3)[(-32) - (-35)]$$

$$A = -1(3) + 2(6) - 3(3)$$

$$A = 0$$

To sum up. The first part of the calculation on row one is show as:

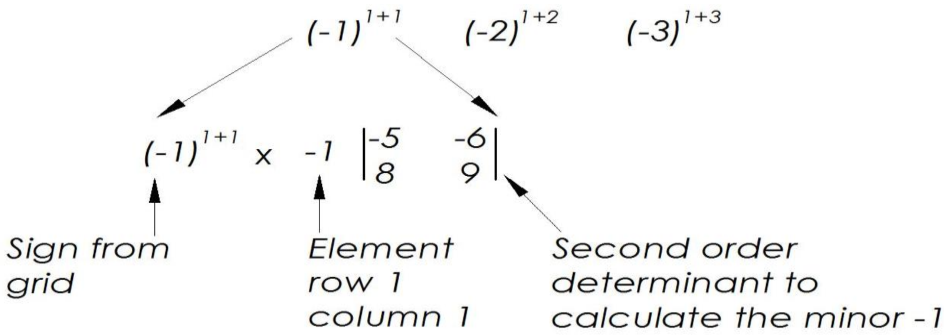


Figure 2.13

2.4 Using Cramer's rule to solve simultaneous equations

Did you know?

Figure 2.14

Gabriel Cramer (1704 – 1752)
Nationality: Swiss
Famous For: Cramer's rule
In 1750 he published Cramer's rule, giving a general formula for the solution for any unknown in a linear equation system having a unique solution, in terms of determinants implied by the system.

2.4.1 Using Cramer's rule to calculate determinants of the second order

As with the other methods of calculating determinants, first manipulate each equation to get it into the standard form:

$$3y - 7x = 16 \quad \text{and} \quad 5x - 10 = 6y$$

Becomes:

$$-7x + 3y = 16 \quad \text{and} \quad 5x - 6y = 10$$

Worked Example 2.6

Use Cramer's rule to solve for x and y :

$$-7x + 3y = 16 \quad \text{and} \quad 5x - 6y = 10$$

Solution:

$$D_0 = \begin{vmatrix} -7 & 3 \\ 5 & -6 \end{vmatrix} = (-7)(-6) - (5)(3) = 27$$

Find the determinant of x: (D_x)

Create the second order determinant with the coefficients of x excluded and replace with the coefficients on the right side of the equation:

$$D_x = \begin{vmatrix} 16 & 3 \\ 10 & -6 \end{vmatrix} = (16)(-6) - (10)(3) = -126$$

Find the determinant of y: (D_y)

Create the second order determinant with the coefficients of y excluded and replace with the coefficients on the right side of the equation:

$$D_y = \begin{vmatrix} -7 & 16 \\ 5 & 10 \end{vmatrix} = (-7)(10) - (5)(16) = -150$$

Apply the rule:

$$x = \frac{D_x}{D_0} = \frac{-126}{27} = 4.67$$

$$y = \frac{D_y}{D_0} = \frac{-150}{27} = 5.56$$

2.4.2 Using Cramer's rule to calculate determinants of the Third order

To solve for three unknowns, three equations are needed. first manipulate each equation to get it into the standard form:

Worked Example 2.7

Rewrite in standard form:

$2x + 3y - Z = 1$

$3x + 5y + 2Z = 8$

$-x + 2y + 3Z = 1$

Write down the matrix:

$$\begin{vmatrix} 2 & 3 & -1 \\ 3 & 5 & 2 \\ -1 & 2 & 3 \end{vmatrix}$$

Find the determinant of the top row:

$$|D| = 2\begin{vmatrix} 5 & 2 \\ 2 & 3 \end{vmatrix}(-3)\begin{vmatrix} 3 & 1 \\ -1 & 3 \end{vmatrix}(-1)\begin{vmatrix} 3 & 5 \\ -1 & 2 \end{vmatrix}$$

$$= 2(11) - 3(11) - 1(11)$$

$$|D| = -22$$

Find the determinant of y: (D_y):

$$|D_y| = \begin{vmatrix} 2 & 1 & -1 \\ 3 & 8 & 2 \\ -1 & 1 & 3 \end{vmatrix}$$

$$|D_y| = 2\begin{vmatrix} 8 & 2 \\ 1 & 3 \end{vmatrix}(-1)\begin{vmatrix} 3 & 2 \\ -1 & 3 \end{vmatrix} + (-1)\begin{vmatrix} 3 & 8 \\ -1 & 1 \end{vmatrix}$$

$$|D_y| = 2(22) - 1(11) - 1(11)$$

$$= -22$$

Apply the rule:

$$y = \frac{D_y}{D} = \frac{22}{-22} = -1$$

Activity 2.1

1. Given:

$$\begin{vmatrix} 2 & 1 & -5 \\ 1 & -1 & 1 \\ 4 & 2 & -3 \end{vmatrix}$$

Determine the following:

a. The determinant of co-efficients.
b. The minor of -5.
c. The co-factor of -1.

2. Given:

$$\begin{vmatrix} 2 & 2 & 1 \\ 1 & -1 & 0 \\ 3 & -3 & 2 \end{vmatrix}$$

Calculate the following:

a. The determinant of the co-efficients.
b. The co-factor of -1.
c. The minor of -3.

3. Given:

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & -1 & -2 \\ -3 & -4 & -5 \end{vmatrix}$$

a. Calculate the value of the determinant.
b. What is the minor of -5?
c. What is the cofactor of -27?

4. Given:
   $2x - 2y - Z = 3$
   $4x + 5y - 2Z = -3$
   $3x + 4y - 3Z = -7$
   a. Solve for the value of Z by using Cramer's rule.
   b. Determine the value of the cofactor of -3.

Answers:

1. -21; 6; 14

2. -8; 1; -1

3. 10; -9; -2

4. -27; -81; $Z = 3$; -6

Activity 2.2

1. Given:
   $3y - x = 2$
   $2y = x$
   Solve for x by using Cramer's Rule.

2. Given:
   $2a + 3b = 14$
   $3a - 2b = -5$
   Solve for a and b using Cramer's Rule.

3. Solve for x by using determinants if:
   $x + y = 10$
   $x - y = 2$

4. Given:
   $a + 4b + 8c = 0$
   $2a - 5b = -2 + 6c$
   $b - 4c = -3$
   Solve for c by the use of Cramer's Rule.

5. Given:
   $a - 3b + c = 4$
   $2a - c = -2$
   $4a - 3b = 0$
   Solve the value of $'a'$ using Cramer's Rule.

Answers:

1. -1; -4; x =4

2. a=1: b=4

3. -2; -12; x=6

4. 74; 41; 0.554

5. a=-2

Self-Check		
I am able to:	**Yes**	**No**
• Solve simultaneous equations with two or three unknowns		
• Calculate determinants and apply Cramer's rule		
• Calculate the minor of a third order determinant		
• Define the co-factor of the minor		
If you have answered 'no' to any of the outcomes listed above, then speak to your facilitator for guidance and further development.		

Module 3

Complex Numbers

Learning Outcomes
On the completion of this module the student must be able to:

- Solve quadratic equations using complex roots
- Convert complex numbers from rectangular to polar form
- Add, subtract, multiply and divide complex numbers by converting to the correct form
- Determining the conjugate and applying it in calculations
- Describe an argand diagram with the argument and modulus
- Apply De Moivre's theorem to products, quotients and powers
- Solve complex equations

3.1 Introduction

A complex number is a number that can be expressed in the form $a + bi$, where a and b are real numbers and i is the imaginary unit, that satisfies the equation $i^2 = -1$. In this expression, a is the real part and b is the imaginary part of the complex number.

3.2 Solving quadratic equations with complex roots

3.2.1 Complex numbers

Complex numbers have two parts to them:

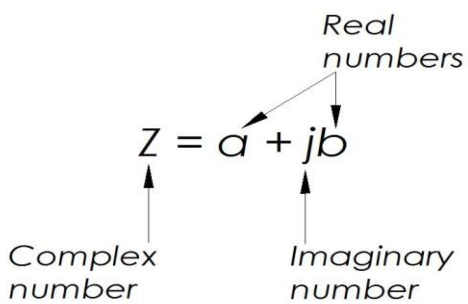


Figure 3.1

There is the real part, a and the imaginary part b. Although b is a real number, it is referred to as the imaginary part because it is always linked to the imaginary.

j, the imaginary number has a value that cannot be written down accurately unless it is denoted as $\sqrt{-1}$.

$$The\ imaginary\ number\ written\ as\ a\ i\ or\ j\ = \sqrt{-1}$$

Using complex numbers in engineering helps get accurate answers to problems that contain the value $\sqrt{-x}$.

An example is shown in the worked example below.

Worked Example 3.1

Find the answer to:

$$\frac{6 \pm \sqrt{-16}}{2}$$

Solution:

$$\frac{6 \pm \sqrt{-16}}{2}$$

$$= \frac{6 \pm \sqrt{-1} \times \sqrt{16}}{2}$$

$$= \frac{6 \pm j \times \sqrt{16}}{2}$$

$$= \frac{6 \pm j \times 4}{2}$$

$$= 3 \pm 2j$$

Note:
It is vital that we can work with square roots. For example:

$$\sqrt{-16} = \sqrt{16} \times \sqrt{-1}$$

The complex number: $Z = 3 \pm 2j$ means $Z = 3 + 2j$ or $Z = 3 - 2j$

3.2.2 The imaginary number

$$j = \sqrt{-1}$$

$$Therefore\ j^2 = \left(\sqrt{-1}\right)^2 = -1$$

Note:

$$\left(\sqrt{x}\right)^2 = (x)^{\frac{2}{2}} = x$$

$$and\ j^3 = \left(\sqrt{-1}\right)^3 = j^2 \times j = (-1)\,(j) = -1$$

$$and\ j^4 = j^2 \times j^2 = (-1)\,(-1) = 1$$

$$and\ j^5 = (j^2)^2 \times j = (-1)^2\,(j) = j$$

Table 3.1 shows the pattern of the first four simplified imaginary numbers and how these are repeated:

Imaginary number	**Value**
$\boldsymbol{j^0}$	**1**
$\boldsymbol{j^1}$	**j**
$\boldsymbol{j^2}$	**-1**
$\boldsymbol{j^3}$	**-j**
j^4	1
j^5	j
j^6	-1
j^7	-j
j^8	1
j^9	j
j^{10}	-1

Table 3.1 imaginary numbers

Worked Example 3.2

Simplify:

1. j^{25}

2. $\frac{36 \pm \sqrt{-40}}{6}$

Solution:

1. $j^{25} = (j)^{24}\, j = (j^2)^{12} \times j = (-1)^{12}\,(j) = j$

2. $\frac{36 \pm \sqrt{-40}}{6}$

$= \frac{36 \pm \sqrt{40}\,.\,\sqrt{-1}}{6}$

$= \frac{36 \pm \sqrt{40}\ j}{6}$

$= 6 \pm 1.054\, j$

3.3 The Argand diagram

3.3.1 The Argand plane

Take a normal Cartesian plane with X and Y co-ordinates:

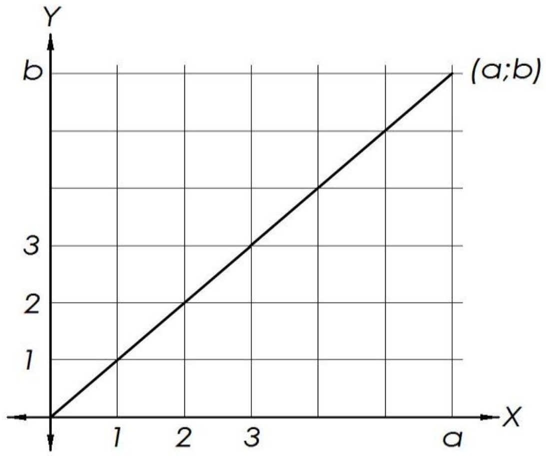


Figure 3.2 Cartesian plane

Did you know?

Figure 3.3

René Descartes (1596-1650)
Nationality: French
Famous For: *Cartesian coordinate system*

The "Cartesian coordinate system" in mathematics is named after Rene Descartes. As a mathematician, he is seen as the father of analytical geometry in addition to explaining "infinitesimal calculus and analysis."

Points along the X axis are:
Point "1" is denoted (1;0) and point "a" is $(a\ ;0)$

Points along the Y axis are:
Point "1" is denoted (0;1) and point "b" is $(0;b)$

For the Argand plane, the X axis denoted real numbers but the Y axis changes to represent imaginary numbers. **Figure 3.4** shows the Argand plane with a vector representing a complex number.

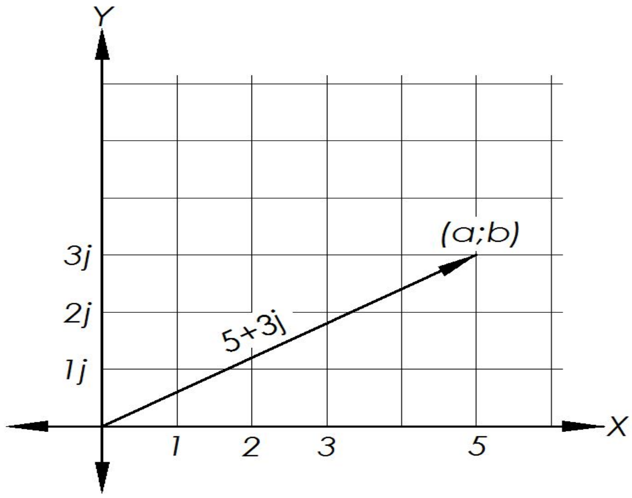


Figure 3.4 Argand plane with vector 5+3j

Note:
The value of j on the Y axis is 1 and the value of 2j is 2 and so on.

Worked Example 3.3

With the help of a graph, write down the geometrical effect of the following:

1. 1
2. 3j-2

Solution:

Multiply the number by *j*, then simplify. Then write in the form (a;b) then make a sketch.

1. $1 \times j = j = (0;1)$

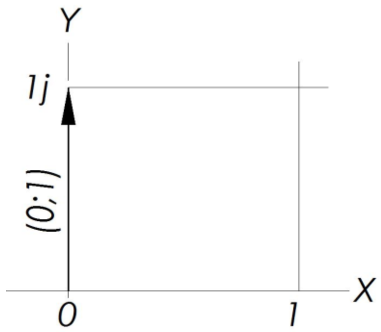


Figure 3.5

2. $j.(3j-2) = 3j^2 - 2j = 3.(-1) - 2j = (-3;-2)$

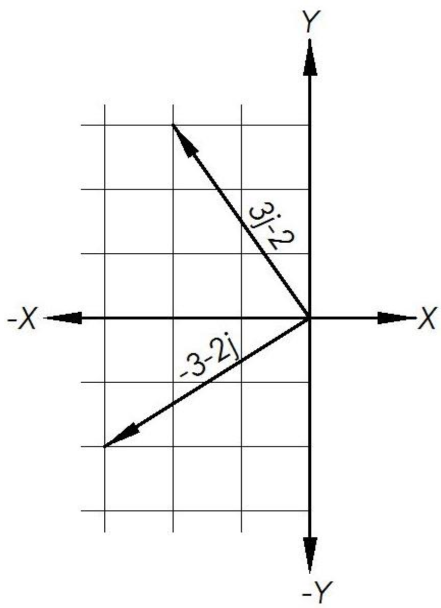


Figure 3.6

Note:
As seen by **Worked example 3.3.2**, any complex number on the Argand plane turns through 90 degrees (anti-clockwise) if it is multiplied by j.

3.3.2 The modulus argument or polar form

A complex number can easily be converted to polar form. Look at **Figure 3.7:**

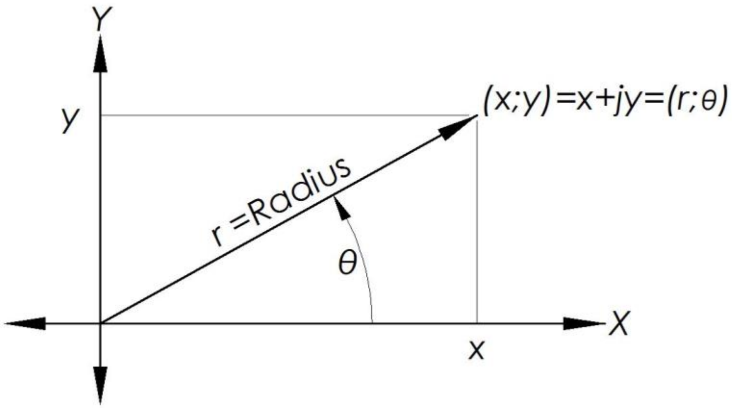


Figure 3.7 polar form of a complex number

From the basic trigonometrical ratios and looking at the graph:

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$\frac{y}{x} = \tan\theta$$

From Pythagoras:

$$r = \sqrt{x^2 + y^2}$$

Worked Example 3.4

Find the polar co-ordinate of $Z = 1 + j$:

Solution:

First create a sketch.

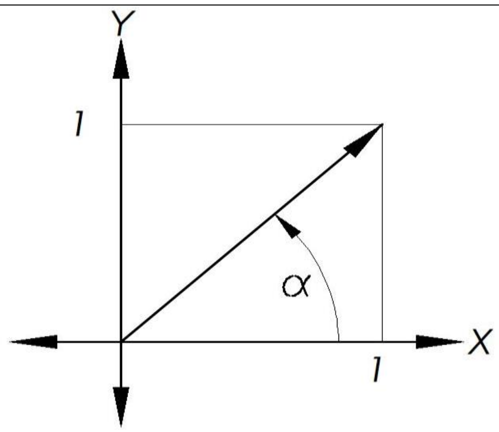


Figure 3.8

$$Auxliary\ angle\ \ \alpha = \tan^{-1}\frac{y}{x} = \ \tan^{-1}\frac{1}{1} = 45°$$

$$But\ \theta = \alpha = 45°$$

$$and\ \ r = \sqrt{x^2 + y^2} = \sqrt{2}$$

$$The\ polar\ for\ is\ written\ \ Z = r\ cis\ \theta\ \ or\ \ Z = r \llcorner\theta$$

$$written\ \ Z = \sqrt{2}\ cis\ 45°\ \ or\ \ Z = \sqrt{2} \llcorner 45°$$

Worked Example 3.5

Write Z=-2+3*j* in cis-form.

Solution:

Note that Z=(-2;3). Create a sketch.

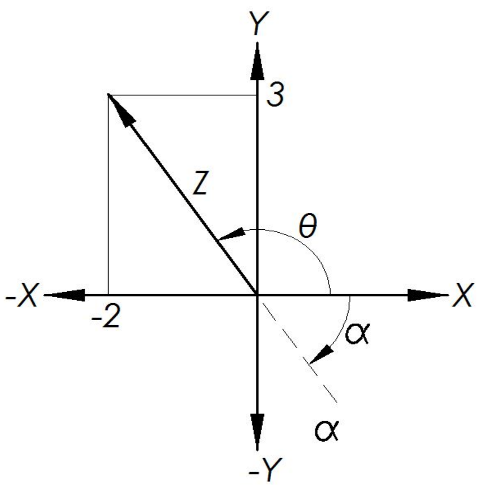


Figure 3.9

$$Auxliary\ angle\ \alpha = \tan^{-1}\frac{3}{2} = 56.309°$$

$$Looking\ at\ the\ sketch\ \theta = 180° - \alpha = 123.69°$$

$$and\ r = \sqrt{(-2)^2 + (3)^2} = \sqrt{13} = 3.61$$

$$Z = 3.61\ cis\ 123.69°$$

3.4 De Moivre's theorem

Did you know?

Figure 3.10

Abraham de Moivre (26 May 1667)
Nationality: French
Famous For: De Moivre's formula

A mathematician known for de Moivre's formula, one of those that link complex numbers and trigonometry, and for his work on the normal distribution and probability theory. He was a friend of Isaac Newton, Edmond Halley, and James Stirling.

De Moivre used the polar form of complex numbers to find the product, quotient and power of complex numbers.

3.4.1 Multiplication of complex numbers

Let $Z_1 = r_1\angle\theta_1\ and\ Z_2 = r_2\angle\theta_2$

$$Z_1 \times Z_2 = r_1 r_2\ \angle\theta_1 + \theta_2$$

3.4.2 Division of complex numbers

$$\frac{Z_1}{Z_2} = \frac{r_1}{r_2} \angle \theta_1 - \theta_2$$

3.4.3 Powers of complex numbers

$$(Z_1)^n = (r_1)^n \angle n\theta_1$$

Worked Example 3.6

If $Z_1 = 2\angle 30°$ *and* $Z_2 = 3\angle 40°$, find $Z_1 \times Z_2$ and $\frac{Z_1}{Z_2}$

Solution:

$$Z_1 \times Z_2 = 2 \times 3 \angle 30° + 40°$$

$$Z_1 \times Z_2 = 6 \angle 70°$$

$$And \quad \frac{Z_1}{Z_2} = \frac{2}{3} \angle 30° - 40°$$

$$\frac{Z_1}{Z_2} = -\frac{2}{3} \angle \underline{-10°}$$

Worked Example 3.7

Write $\frac{1}{(1-j)^3}$ in rectangular form

Solution:

Take the content of the bracket $(1 - j)$ and let that = Z

$$Then \ \frac{1}{(1-j)^3} = \frac{1}{Z^3}$$

$$(1 - j)^{-3} = Z^{-3}$$

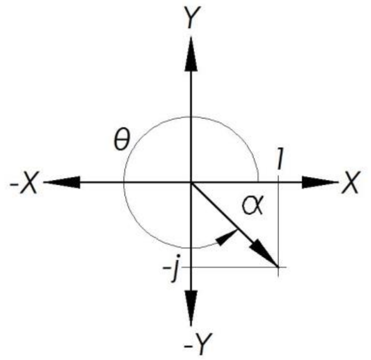


Figure 3.11

$$From\ the\ sketch\ \ r = \sqrt{2}\ and\ \theta = -45$$

$$Write\ in\ polar\ form\ \ Z^{-3} = \left(\sqrt{2}\,\llcorner -45°\right)^{-3}$$

$$Using\ De\ Moivre'sthird\ formula\ \ (Z_1)^n = (r_1)^n\ \llcorner n\theta_1$$

$$= \left(\sqrt{2}\right)^{-3} \llcorner -3 \times 45°$$

$$= 0.3535\ \llcorner 135°$$

$$In\ rectangular\ form\ = -\frac{1}{4} + \frac{1}{4}j$$

3.5 The conjugate of a complex number

Definition:
The conjugate of the complex number $x + jy$ is $x - jy$
The conjugate of Z is indicated as $\bar{Z}$

3.6 Complex equations

Worked Example 3.8

Solve for x if: $2x^2 - 6x + 5 = 0$

Solution:

$x = \frac{-(-6)\pm\sqrt{(-6)^2-4(2)(5)}}{4}$

$x = \frac{6\pm j\sqrt{4}}{4}$

$x_1 = \frac{6+j2}{4}$ and $x_2 = \frac{6-j2}{4}$

$x_1 = 1{,}5 + j0{,}5$ $x_2 = 1{,}5 - j0{,}5$

Activity 3.1

Simplify:

1. j^6
2. j^7
3. j^8
4. j^9

Simplify:

5. $\frac{5 \pm \sqrt{-17}}{5}$
6. $\frac{-12 \pm \sqrt{-20}}{5}$

Answers:

1. -1
2. –j
3. 1
4. J
5. $1 \pm 0.825j$
6. $-2.4 \pm 0.894j$

Activity 3.2

1. Represent the following complex numbers on the Argand diagram:
   1.1 $Z_1 = 2 + j5$
   1.2 $Z_3 = -3 - j$
   1.3 $Z_5 = -j$

2. Investigate the geometrical effect of the product of *j* with:
   2.1 $-j$
   2.2 $9 + 5j$

Answers:

Question 1:

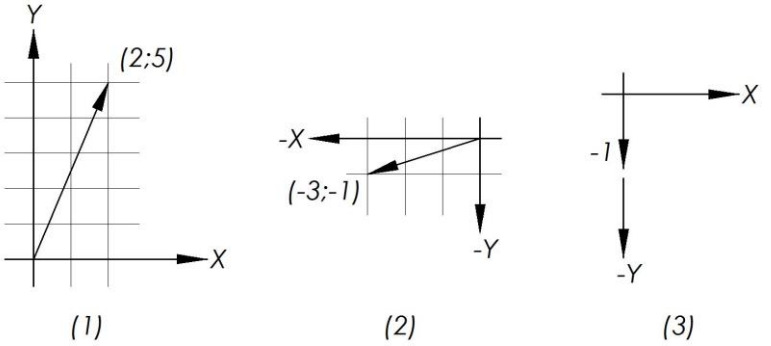


Figure 3.12

Question 2:

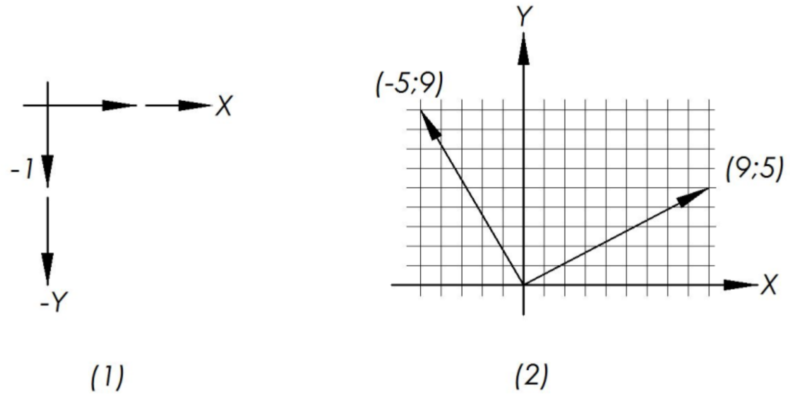


Figure 3.13

Activity 3.3

Find the polar form of the following:

1. $(1) - 3 + 4j$
2. $(3) - 2 - 2j$
3. $4 - j2$
4. j
5. $-1 - j$
6. $-j5$

Find the rectangular form of the following:

7. $3\ cis\ 32^\circ$

8. $4\underline{|36^\circ}$

9. $3\underline{|90^\circ}$

10. $\underline{|-145^\circ}$

11. $3{,}5\ \underline{|3x/2}$

Answers:

1. $(1)5\underline{|126{,}870^\circ}$

2. $(3)\sqrt{8}\underline{|-135^\circ}$

3. $(5)\sqrt{20}\underline{|-26{,}565^\circ}$

4. $(7)\underline{|90^\circ}$

5. $(9)\sqrt{2}\underline{|-135^\circ}$

6. $(11)5\underline{|-90^\circ}$

7. $2{,}54 + j1{,}59$

8. $3{,}24 + j2{,}35$

9. $(5)j3$

10. $-0{,}82 - j0{,}57$

11. (9) $-3{,}5j$

Activity 3.4

Simplify:

1. $(2\underline{|30^\circ})(3\underline{|60^\circ})$
2. $(\sqrt{2}\underline{|40^\circ})(\sqrt{8}\underline{|60^\circ})$

Simplify but give the answer in rectangular form:

3. $\frac{(4-j2)(2-j7)}{4+j6}$
4. $\frac{3-j4}{(2+j)(3-j2)}$

Find the value of:

5. $(2\angle 90^\circ)^4$
6. $(8\angle 42^\circ)^2$

Answers:

1. $6\angle 90^\circ$

2. $4\angle 100^\circ$

3. $-4{,}15 - j1{,}77$

4. $0{,}43 - j - .45$

5. 16

6. $64\angle 84^\circ$

Activity 3.5

What is the conjugate of:

1. $3 - 2j$
2. $c + jd$

If $z = 5 + 6j$, determine:

3. $\bar{z}$
4. $\bar{\bar{Z}}$

Answer to activity 3.5

1. $3 - 2j$

2. $c - jd$

3. $5 - 6j$

4. $5 - 6j$

Activity 3.6

1. Given: $Z = -3 + j2$
   1.1 Convert Z into polar form. Show ALL the steps. θ may only be positive.
   1.2 Represent Z and ALL calculated values on the Argand diagram.

2. Given:
   $Z = -2 + j5$
   2.1 Find $\overline{Z}$
   2.2 Convert Z into polar form. Show ALL steps. θ may only be positive.
   2.3 Represent Z and ALL calculated values in 2.2 on the Argand diagram.

3. Solve for x and y if:
   $$x + jy = \frac{(3 + j5)(2 - j7)}{(1 - j3)}$$

4. Given:
   $Z = -3 + j2$
   4.1 Find $\overline{Z}$
   4.2 Convert Z into polar form. θ must be positive. Show ALL steps
   4.3 Represent all the calculated values in 4.2 on the Argand diagram.

5. Given: $Z_1 = 20\underline{|63°}$ and $Z_2 = 4\underline{|36°}$, determine $\left(\frac{Z_1}{Z_2}\right)^3$. Show ALL steps. Leave the answer in rectangular form.

6. Solve for x and y if:
   $$(3 - 4j)^2 = \frac{-x - yj}{j^2}$$

7. Solve for x if:
   $x + y + yj + 2j = 8j + 7$

8. Given:
   $Z_1 = 1 - j1{,}732$
   $Z_2 = -6{,}928 - j4$
   8.1 Convert Z_1 and Z_2 into polar form. The arguments may only be positive.
   8.2 Calculate $\frac{Z_1}{Z_2}$ and leave the answer in $a + jb$ form.

9. Given:

$Z_1 = \left(4\underline{|60°}\right)^3$

$Z_2 = \left(3\underline{|-30°}\right)^3$

$Z_3 = \left(2\underline{|-150°}\right)^4$

Find: $\frac{Z_1 \times Z_2}{Z_3}$

Answers:

1. $\therefore Z = \sqrt{13}\underline{|146{,}296°}$

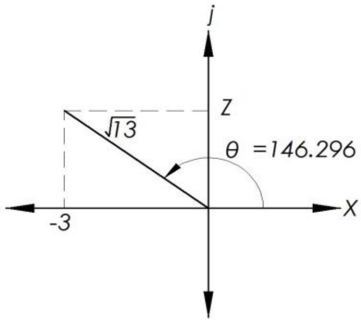


Figure 3.14

2. $\bar{Z} = -2 + j5$

$r = \sqrt{(-2)^2 + (-5)^2}$

$Z = \sqrt{29}\,\underline{|248{,}199°}$

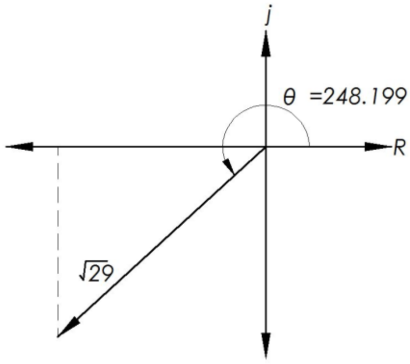


Figure 3.15

3. $x = \frac{74}{10}$ and $y = \frac{112}{10}$

4. $\bar{Z} = -3 - j2$

$\bar{Z} = 3{,}606\underline{|213{,}69°}$

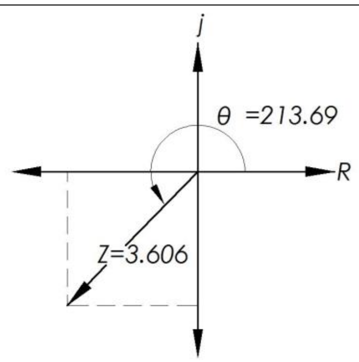


Figure 3.16

5. $= 19{,}554 + j123{,}461$

6. $x = -7; y = -24$

7. $Z_1 = 2|\underline{300^\circ}$

   $Z_2 = 8|\underline{210^\circ}$

   $= 0 + j0{,}25$

8. $\therefore x = 1$

9. $= 31{,}177 - j18$

Self-Check

I am able to:	**Yes**	**No**
• Solve quadratic equations using complex roots		
• Convert complex numbers from rectangular to polar form		
• Add, subtract, multiply and divide complex numbers by converting to the correct form		
• Determining the conjugate and applying it in calculations		
• Describe an argand diagram with the argument and modulus		
• Apply De Moivre's theorem to products, quotients and powers		
• Solve complex equations		
If you have answered 'no' to any of the outcomes listed above, then speak to your facilitator for guidance and further development.		

Module 4

Trigonometry

Learning Outcomes
On the completion of this module the student must be able to:

- Describe the concept of positive and negative angles
- Apply the identities of trigonometric ratios to solve equations
- Derive the identities for trigonometric ratios
- Derive the co-ratios from trigonometric identities and apply them

4.1 Introduction

Trigonometry comes from the Greek words for "triangle" and "measure" and it is a branch of mathematics that studies relationships involving lengths and angles of triangles.

4.2 Basic trigonometrical identities

Definition:
An identity differs from an equation. When we solve an equation, the unknown is valid for only certain values. With identities, the unknown is valid for all values.
For example the equation: $\sin^2\theta = 1 \quad \theta$ *can only be* 90^o.
But the identity: $\sin^2\theta + \cos^2\theta = 1 \quad$ *is valid for all values of* θ

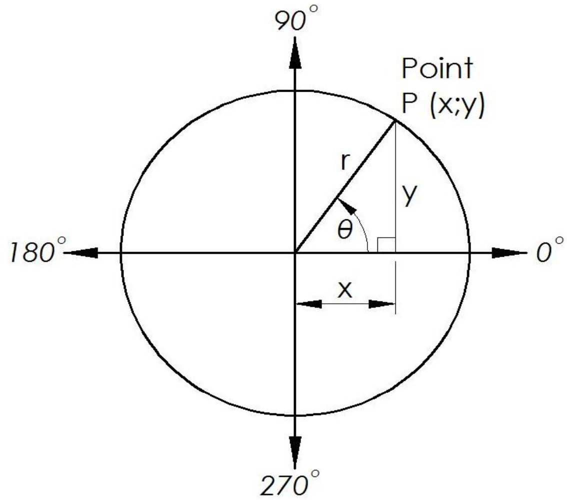


Figure 4.1 Trigonometric ratio

4.2.1 Trigonometric functions

Referring to the above Cartesian plane with the point P rotating anticlockwise, the basic functions are:

$$\sin\theta \;=\frac{y}{r} \quad and \quad \text{cosec}\frac{r}{y}$$

$$\cos\theta=\frac{x}{r} \quad and \quad \sec\frac{r}{x}$$

$$\tan\theta=\frac{y}{x} \quad and \quad \text{cosec}\frac{x}{y}$$

4.2.2 Basic identities

We can use these functions to develop these useful identities:

$$\tan\theta=\frac{\sin\theta}{\cos\theta} \quad and \quad \cot\theta=\frac{\cos\theta}{\sin\theta}$$

And the theorem of Pythagoras is important.

Did you know?

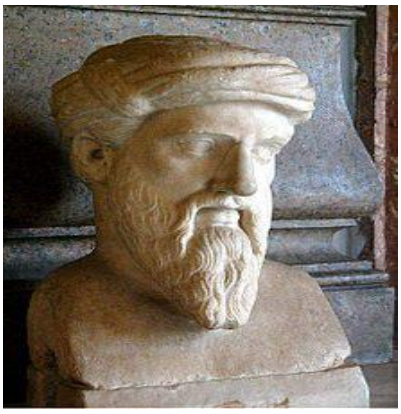


Figure 4.2

Pythagoras of Samos (**570 – 495 BC)**
Nationality: Greek
Famous For: *Pythagoras's theorem*

Pythagoras is often described as the first pure mathematician. He is an extremely important figure in the development of mathematics yet we know relatively little about his mathematical achievements.

$$From\ the\ Cartesian\ plane \quad x^2+y^2=r^2$$

$$Divide\ by\ r^2 \quad \left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1$$

$$\cos^2\theta + sin^2\theta = 1 \quad \dots \quad (1)$$

From (1) it follows:

$$sin^2\theta = 1 - \cos^2\theta \quad \therefore \quad \sin\theta = \pm\sqrt{1 - \cos^2\theta}$$

$$cos^2\theta = 1 - \sin^2\theta \quad \therefore \quad \cos\theta = \pm\sqrt{1 - \sin^2\theta}$$

Divide (1) by $cos^2\theta$:

$$1 + tan^2\theta = sec^2\theta \quad \dots \quad (2)$$

Divide (1) by $sin^2\theta$:

$$cot^2\theta + 1 = cosec^2\theta \quad \dots \quad (3)$$

From (1) and (2) it follows:

$$\sec\theta = \pm\sqrt{1 + tan^2\theta}$$

$$\tan\theta = \pm\sqrt{sec^2\theta - 1}$$

$$\mathrm{cosec}\,\theta = \pm\sqrt{cot^2\theta + 1}$$

$$\cot\theta = \pm\sqrt{cosec^2\theta - 1}$$

4.2.3 Identities with compound angles

$$\sin A + B = \sin A\cos B + \cos A\sin B \quad \dots \quad (4)$$

$$\sin A - B = \sin A\cos B - \cos A\sin B \quad \dots \quad (5)$$

$$\cos A + B = \cos A\cos B - \sin A\sin B \quad \dots \quad (6)$$

$$\cos A - B = \cos A\cos B + \sin A\sin B \quad \dots \quad (7)$$

$$\tan A + B = \frac{\tan A + \tan B}{1 - \tan A\tan B} \quad \dots \quad (8)$$

$$\tan A - B = \frac{\tan A - \tan B}{1 + \tan A\tan B} \quad \dots \quad (9)$$

4.2.4 Identities with double angles

When the two angles are the same ie. $A = B$, then from (4),(6) and (8) we get:

$$\sin 2A = 2\sin A\cos A \quad \ldots \quad (10)$$

$$\cos 2A = cos^2 A - sin^2 A \quad \ldots \quad (11)$$

$$\tan 2A = \frac{2\tan A}{1 - tan^2 A} \quad \ldots \quad (12)$$

From (11):

$$sin^2 A = \frac{1}{2}(1 - cos^2 2A) \quad \ldots \quad (13)$$

Replacing $sin^2 A$ with $1 - cos^2 2A$ in (11) we find:

$$\cos 2A = cos^2 A - 1 - cos^2 2A$$

$$\cos 2A = cos^2 A - (1 - cos^2 2A)$$

$$\cos^2 A = \frac{1}{2}(1 + \cos 2A) \quad \ldots \quad (14)$$

Worked Example 4.1

$$Simplify\ \sec\theta - \sec\theta\ \sin^2\theta$$

Solution:

$\sec\theta - \sec\theta\ \sin^2\theta \ = \ \sec\theta\ (1 - \sin^2\theta)$

$= \sec\theta\ (cos^2\theta)$

$= \frac{1}{\cos\theta}\ (cos^2\theta)$

$= \cos\theta$

Worked Example 4.2

Prove that:

$$\sin^2 x\cot x\sec x + \cos^2 x\tan x\,\text{cosec}\, x = \sin x + \cos x$$

Solution:

$$\text{Take the LHS} = \sin^2 x\cot x\sec x + \cos^2 x\tan x\,\text{cosec}\, x$$

$$= \sin^2 x \times \frac{\cos x}{\sin x} \times \frac{1}{\cos x} + \cos^2 x \times \frac{\sin x}{\cos x} \times \frac{1}{\sin x}$$

$$= \sin x + \cos x = RHS$$

Worked Example 4.3

Show that:

$$\frac{1 + \sin\theta}{\cos\theta} = \frac{\cos\theta}{1 - \sin\theta}$$

Solution:

$$LHS = \frac{1+\sin\theta}{\cos\theta}$$

$$= \frac{1+\sin\theta}{\cos\theta} \times \frac{1-\sin\theta}{1-\sin\theta} \quad It's\ like\ multiplying\ by\ 1$$

$$= \frac{1-\sin^2\theta}{\cos\theta\,(1-\sin\theta)}$$

$$= \frac{cos^2\,\theta}{\cos\theta\,(1-\sin\theta)}$$

$$= \frac{\cos\theta}{(1-\sin\theta)} = RHS$$

4.2.5 Identities with half angles

The following half angle identities are shown below:

$$\sin\theta = sin\left(\frac{\theta}{2} + \frac{\theta}{2}\right)$$

$$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} \quad \ldots \quad (15)$$

$$\cos\theta = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} \quad \ldots \quad (16)$$

$$\cos\theta = 2\ \cos^2\ \frac{\theta}{2} \quad \ldots \quad (17)$$

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{\cos\theta+1}{2}} \quad \ldots \quad (18)$$

$$\cos\theta = 1 - 2\ sin^2\ \frac{\theta}{2} \quad \ldots \quad (19)$$

$$\text{Sin}\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}} \quad \ldots \quad (20)$$

4.3 Trigonometric equations

4.3.1 Sum and difference identities

Worked Example 4.4

Write $\sin A + \sin B$ as a product of trigonometric functions.

Solution:

$$Let \quad A = x + y \quad and \quad B = x - y$$

$$Then \quad A + B = 2x \quad \ldots \quad (\$)$$

$$and \quad A - B = 2y \quad \ldots \quad (\$\$)$$

Then it follows:

$$\begin{aligned} \sin A + \sin B &= \sin(x+y) + \sin(x-y) \\ &= (\sin x \cos y + \cos x \cos y) + (\sin x \cos y - \cos x \sin y) \\ &= 2 \sin x \cos y \end{aligned}$$

Then from ($) and ($$):

$$x = \frac{A+B}{2} \quad and \quad y = \frac{A-B}{2}$$

But:

$$\sin A + \sin B = 2 \sin x \cos y$$

$$\therefore \quad \sin A + \sin B = 2 \sin \tfrac{A+B}{2} \cos \tfrac{A-B}{2} \quad \ldots \quad (21)$$

It can also be shown:

$$\sin A - \sin B = 2 \cos \frac{A+B}{2} \sin \frac{A-B}{2} \quad \ldots \quad (22)$$

$$\cos A + \cos B = 2 \cos \frac{A+B}{2} \cos \frac{A-B}{2} \quad \ldots \quad (23)$$

$$\cos A - \cos B = 2 \sin \frac{A+B}{2} \sin \frac{A-B}{2} \quad \ldots \quad (24)$$

Worked Example 4.5

Write $\cos A \cos B$ as a sum of trigonometric functions.

Solution:

Add the following two identities:

$$\cos A + B = \cos A \cos B - \sin A \sin B \quad \ldots \quad (6)$$

$$\cos A - B = \cos A \cos B + \sin A \sin B \quad \ldots \quad (7)$$

$$\cos(A + B) + \cos(A - B) = 2\cos A \cos B \quad \ldots \quad (6) + (7)$$

$$2\cos A \cos B = \cos(A + B) + \cos(A - B)$$

$$\cos A \cos B = \frac{1}{2}[\cos(A + B) + \cos(A - B)] \quad \ldots \quad (25)$$

Similarly:

$$\cos A \sin B = \frac{1}{2}[\sin(A + B) - \sin(A - B)] \quad \ldots \quad (26)$$

$$\sin A \sin B = -\frac{1}{2}[\cos(A + B) - \cos(A - B)] \quad \ldots \quad (27)$$

$$\sin A \cos B = -\frac{1}{2}[\sin(A + B) - \sin(A - B)] \quad \ldots \quad (28)$$

Worked Example 4.6

Write the following sum as a product:

$$\sin 7\theta + \sin 3\theta$$

Solution:

$$\sin 7\theta + \sin 3\theta = 2\sin 5\theta \cos 2\theta \quad [according\ to\ (21)]$$

4.3.2 Positive angles

When working out equations, the unknown will be an angle or angles. Inverse trigonometric functions will be used.

Figure 4.3 Greek stamp in honour of Hipparchus

Did you know?

Hipparchus of Nicaea (190 – 120 BC)
Nationality: Greek
Famous For: *founder* ***of trigonometry***

He was an astronomer, geographer, and mathematician. He is considered the founder of trigonometry but is most famous for his incidental discovery of precession of the equinoxes.

Your answer will be limited to values of a closed interval, for example between 0° and 360°.

$$\text{Written in this way} \quad 0° \leq A \leq 360°$$

This will change a relation like $\sin A = \frac{1}{2}$ where the answer may be:

$$A = 30° \ or \ A = 150° \ or \ A = -210° \ \text{and so on to infinity}$$

It will rather be a function that may have one or two answers.

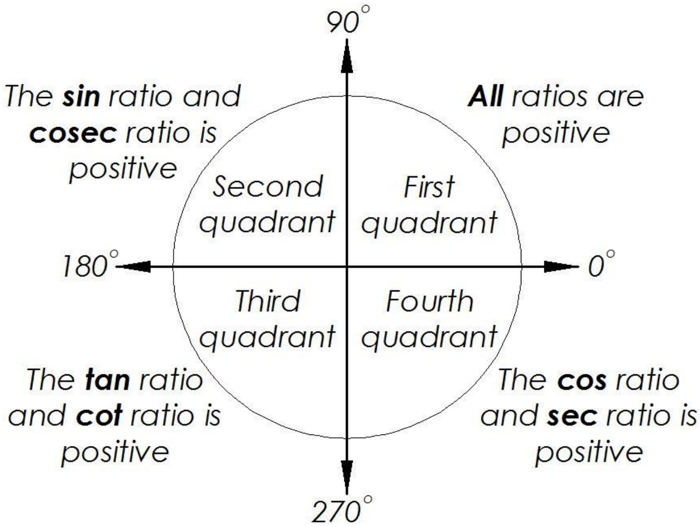
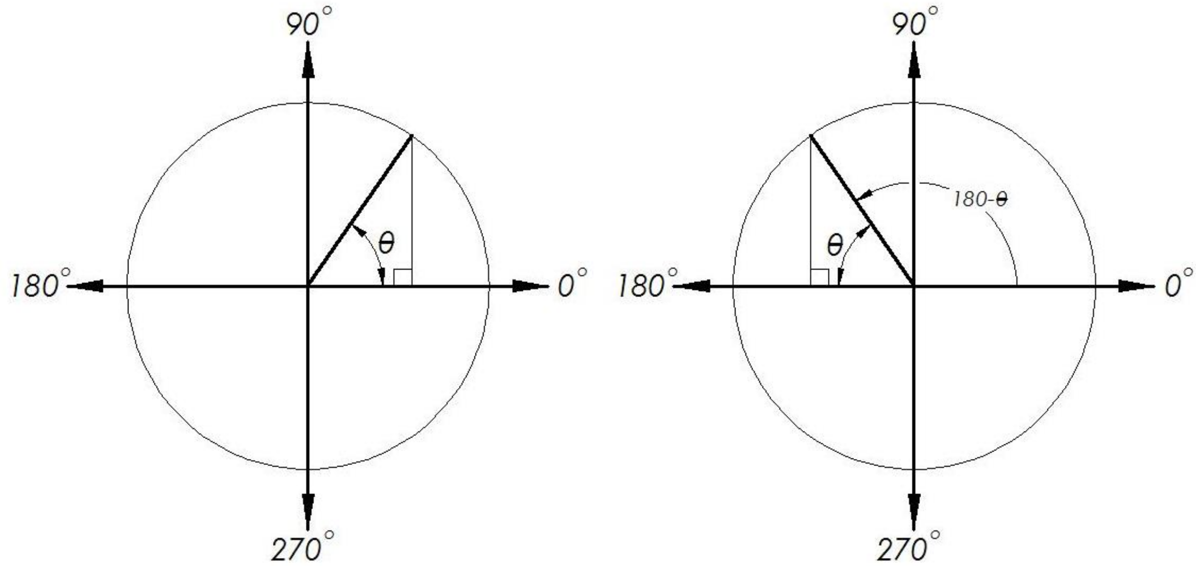


Figure 4.4 The four quadrants going anticlockwise

The angle θ, must be an acute angle. Looking at **Figure 4.5** It is clear that this is not a problem in the first quadrant as the angle subtends from the right hand horizontal which is 0° and rotates anticlockwise. So in the first quadrant, θ is between $0°$ and $90°$.

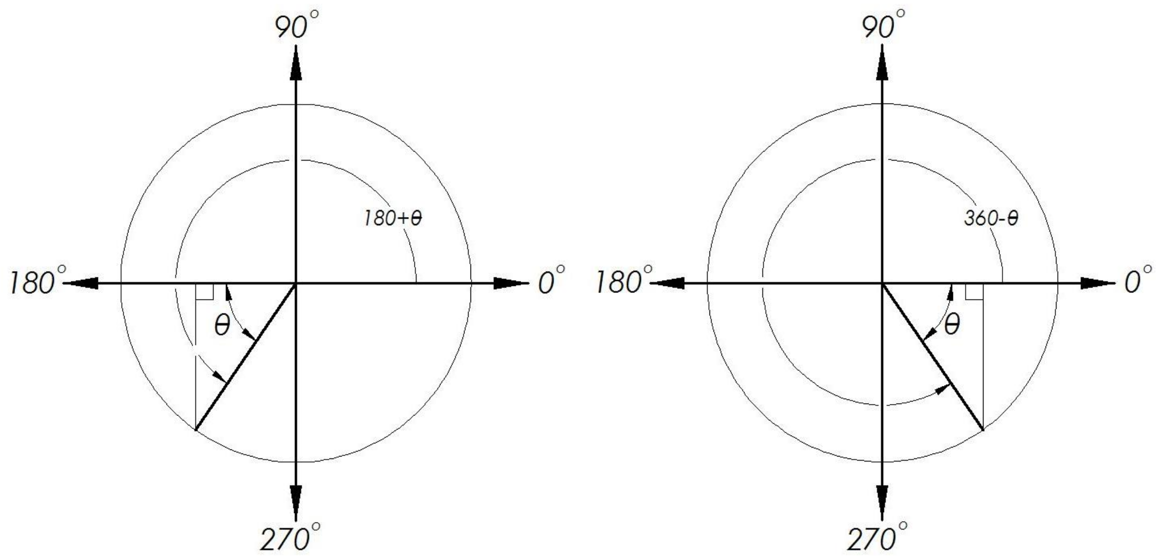


First quadrant Second quadrant

Figure 4.5 The first and second quadrants

Notice that in **Figure 4.5**, the reference angle θ must be an acute angle. This means that the solution to the ratio is $(180° - \theta)$.

In the third quadrant shown in **Figure 4.6**, the reference angle θ remains an acute angle. This means that the solution to the ratio is $(180° + \theta)$.



Third quadrant Fourth quadrant

Figure 4.6 The third and fourth quadrants

And in the fourth quadrant shown in **Figure 4.6**, the reference angle θ remains an acute angle. This means that the solution to the ratio is $(360° - \theta)$.

4.3.3 Solving equations

Worked Example 4.7

Solve for θ:

1. $\cos\theta = -0.6 \quad for \quad \theta \in [0°; 360°]$

2. $2\sin^3\theta - \sin\theta = 0\,; \quad -90° \leq \theta \leq 90°$

Solution:

1. $\cos\theta = -0.6$

Find the acute angle: $\cos\theta = -0.6$

$$\theta = \cos^{-1} -0.6 = 53.13°$$

This acute angle is not the solution because the minus sign (-0.6) shows that the solution lies in the 2nd and 3rd quadrants. So the next step is always to check which quadrants the solution lies in by looking at the sign. See **Figure 4.7.**

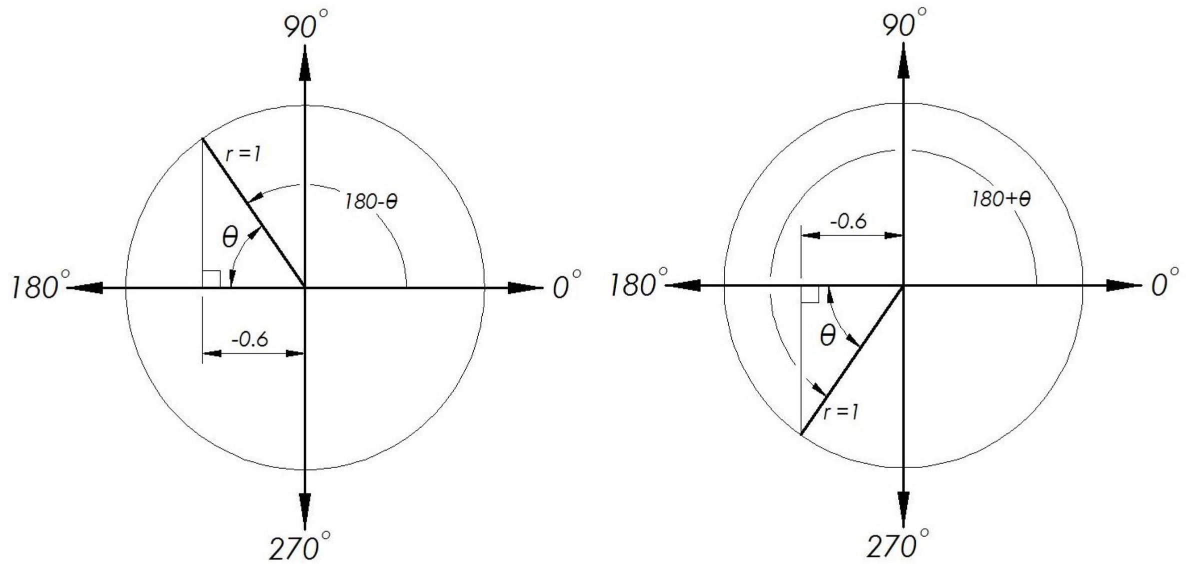


Figure 4.7 The cos θ is negative in the second and third quadrants

$$\therefore \ The\ solution = 180 - \theta \qquad or \qquad 180 + \theta$$

$$= 180 - 53.13° \quad or \quad 180 + 53.13°$$

$$= 126.87° \quad or \quad 233.13°$$

2. $2\sin^3\theta - \sin\theta = 0$

$$\sin\theta\ (2\sin^2\theta - 1) = 0$$

$$\sin\theta = 0 \quad or \quad 2\sin^2\theta - 1 = 0$$

$$\sin\theta = 0 \quad or \quad \sin\theta = \sqrt{\frac{1}{2}}$$

$$\sin\theta = 0 \quad or \quad \sin\theta = 0.707$$

$$\sin\theta = 0 \quad or \quad \sin\theta = 0.707$$

$$\theta = 0° \quad or \quad \theta = 45°$$

Worked Example 4.8

If $\sin x = \frac{12}{13}$ and $\cos y = \frac{8}{17}$ and both x and y are acute angles, calculate WITHOUT the use of a calculator, the value of $\sin(x + y)$.

Solution:

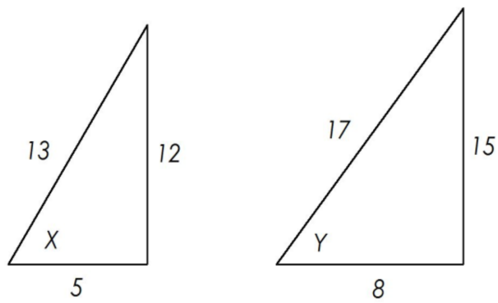


Figure 4.8

$$\sin(x + y) = \sin x \cos y + \sin y \cos x$$

$$= \frac{12}{13} \times \frac{8}{17} + \frac{15}{17} \times \frac{5}{13}$$

$$= \frac{48}{221} + \frac{75}{221}$$

$$= \frac{123}{221}$$

Worked Example 4.9

Derive a formula for $\cos 2B$ in terms of $\sin B$.

Solution:

$$\cos 2B = \cos(B + B)$$

$$= \cos^2 B - \sin^2 B$$

$$= (1 - \sin^2 B) - \sin^2 B$$

$$\cos 2B = 1 - 2\sin^2 B$$

Worked Example 4.10

Calculate WITHOUT the use of a calculator, the value of $cosec\ 75°$.

Solution:

$$\operatorname{cosec} 75° = \frac{1}{\sin 75°}$$

$$\sin 75° = \sin 45° \cos 30° + \sin 30° \cos 45°$$

$$\sin 75° = \frac{1}{\sqrt{2}} . \frac{\sqrt{3}}{2} + \frac{1}{2} . \frac{1}{\sqrt{2}}$$

$$\sin 75° = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

$$\therefore \quad \operatorname{cosec} 75° = \frac{2\sqrt{2}}{\sqrt{3}+1}$$

Worked Example 4.11

Prove that $\cos 2A = \frac{7}{25}$ if $\sin A = \frac{3}{5}$.

Solution:

$$\begin{aligned}\cos 2A &= 1 - 2\sin^2 A \\ &= 1 - 2\left(\frac{3}{5}\right)^2 \\ &= 1 - \frac{18}{25} \\ \cos 2A &= \frac{7}{25}\end{aligned}$$

Activity 4.1

Prove the following identities:

1. $cosec\ B.\cot B + cosec^2 B = \frac{1}{1-\cos B}$
2. $(\sin B + \cos B)^2 = 1 + 2\sin B\cos B$
3. $\frac{\sin 2x}{\cos 2x - 1} = -\cot x$
4. $\frac{2\sin y + 1}{\sin 2 + \cos y} = \sec y$
5. $\frac{Sin2A - Cos2A + 1}{Sin2A + Cos2A + 1} = \tan A$
6. $cos^2 x sec^2 x - cos^2 x = sin^2 x$
7. $\tan A + \cot A = \sec A\ cosec\ A$
8. $tan^2 x - sec^2 x = -1$
9. $\frac{\sin\theta}{1+\cos\theta} + \frac{1+\cos\theta}{\sin\theta} = 2\ cosec\ \theta$

Answers:
With these answers, the LHS must equal the RHS

Activity 4.2

1. Solve for x if:
   $4sin^2x - 5\cos x = 2\ ; 0° \leq x \leq 360°$

2. Solve for B if:
   $3sin^2B - 2\sin B = 1\ \ and\ 0° \leq B \leq 360°$

3. Solve for θ if:
   $sec^2\theta + \tan\theta - 3 = 0\ ;\ 0° \leq \theta \leq 360°$

4. Solve for P if:
   $4sin^2P - 1 = 0\ and\ 0° \leq P \leq 360°$

5. Solve for A if:
   $2cos^2A - \text{Cos}\,A; 0° \leq A \leq 360°$

6. Solve for θ if:
   $12cot^2\theta + 5\cot\theta - 2;\ 0° \leq \theta \leq 360°$

7. Solve for θ if:
   $6cos^2\theta - 4 = 0\ ;\ \theta \in [0°; 180°]$

8. Solve for P if:
   $sin^3\theta - cs^3\theta = 0°;\ \theta\epsilon\left[-\frac{3x}{2}; 0\right]$

Answers:

1. $x_1 = 108{,}602°$
   $x_2 = 251{,}397°$

2. $B_1 = 71{,}565$
   $B_2 = 108{,}435$
   $B_3 = 288{,}435$
   $B_4 = 108{,}435$

3. $\theta = 116{,}565°$ $\quad\theta = 45°$
   $\theta = 296{,}565°$ $\quad\theta = 225°$

4. $P_1 = 30°$ $\quad P_3 = 210°$
   $P_2 = 150°$ $\quad P_4 = 330°$

5. $A = 120°$ and $240°$

6. $\theta_1 = 75{,}964°$ $\quad\theta_3 = 123{,}69°$
   $\theta_2 = 255{,}964°$ $\quad\theta_4 = 303{,}69°$

7. $\theta = 35{,}26°$ $\theta = 144{,}74°$

8. $\theta = \frac{3\pi}{4}$

Activity 4.3

1. If $\sin A = \frac{1}{2}$ and $\sin B = \frac{1}{\sqrt{2}}$, and both A and B are acute angles, calculate, WITHOUT using a calculator, the value of $\cos(A - B)$.
2. Simplify:
   $$\frac{2\cot x}{1 + cot^2 x}$$
3. Derive a formula for $\sin 2A$.
4. Derive a formula for $\cos\frac{A}{2}$ if $\cos 2A = 2\,cos^2 A - 1$
5. Calculate the value of $\sin 120°$ without the use of a calculator.
6. Determine the value of $\cos 105°$ without the use of a calculator.
7. Simplify:
   $\cos ecy + \cot y$
8. If $\sin P = \frac{12}{13}$ and $\cos Q = \frac{3}{5}$, and both P and Q are acute angles, determine without the use of a calculator, the value of $\cos(P + Q)$.
9. If $\sin A = 0{,}6$ and $\sin B = 0{,}4$; and both A and B are acute angles, determine the value of $\text{Cos}(A + B)$ without the use of a calculator.
10. Simplify:
    $$\frac{1 + cos^2 x}{\cot x \cos ecx}$$
11. Determine $cos^2 45° - sin^2 45°$ WITHOUT the use of a calculator.
12. If $\cos x = \frac{3}{5}$ and x is an acute angle, determine the following WITHOUT using a calculator.
    12.1 $\sin 2x$
    12.2 $\cos 2x$

Answers:

1. $\frac{\sqrt{3}+1}{2\sqrt{2}}$

2. $\sin 2x$

3. $2\sin A\cos A$

4. $\cos\frac{A}{2} = \sqrt{\frac{1}{2}+\frac{\cos A}{2}}$

5. $\frac{\sqrt{3}}{4}$

6. $\frac{1-\sqrt{3}}{2\sqrt{2}}$

7. $\frac{1+\cos y}{\sin y}$

8. $\frac{-33}{65}$

9. $\frac{-4\sqrt{21}-6}{25}$

10. $\sec x$

11. 0

12.

    12.1 $\frac{24}{25}$

    12.2 $-\frac{7}{25}$

Self-Check

I am able to:	Yes	No
• Describe the concept of positive and negative angles		
• Apply the identities of trigonometric ratios to solve equations		
• Derive the identities for trigonometric ratios		
• Derive the co-ratios from trigonometric identities and apply them		
If you have answered 'no' to any of the outcomes listed above, then speak to your facilitator for guidance and further development.		

Module 5

Sketch Graphs

Learning Outcomes
On the completion of this module the student must be able to:

- Identify domain and range, dependent and independent variables, functions and relations, points of symmetry, continuous and discontinuous functions, inverse functions and relations
- Describe and draw graphs without tables of values or point for point plotting

5.1 Introduction

In mathematics, the graph of a function *f* is the collection of all ordered pairs $(x, f(x))$. If the function input x is a scalar, the graph is a two-dimensional graph, and for a continuous function is a curve.

5.1.1 System of axes and ordered pairs

The reason why the three numbered pairs in **Figure 5.1** are termed ordered pairs is because the x value is written first and then the y value.

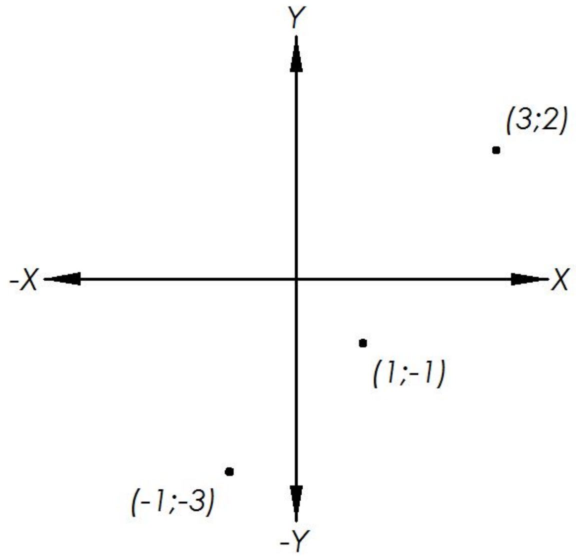


Figure 5.1 A system of axes with three orders pairs

5.1.2 Domain and range

The x co-ordinates in ordered pairs are the domain and the y co-ordinates are the range. In **Figure 5.1**, the domain is (3; 1; -1) and the range is (2; -1; -3).

5.1.3 dependent and independent variables

The following equation has an independent variable x and a dependent variable y.

$$y = \frac{2}{3}x - 1$$

Values for x are chosen and when inserted into the equation, y is worked out and changes depending on the value of x. So y is dependent on x.

5.1.4 Functions

A function is symbolically written as: $f: x \rightarrow y$ and it reads as:

Function f from x to y .

$f(x)$ represents y so the ordered pair $(x; y) = (x; f(x))$

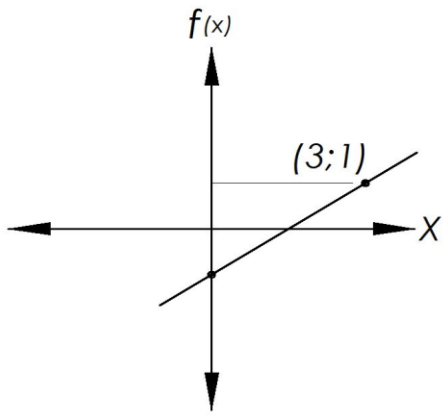


Figure 5.2 A system of axes with one orders pair

Take the equation: $f(x) = \frac{2}{3}x - 1$

Looking at **Figure 5.2**, choose a value of 3 for x:

$$f(3) = \frac{2}{3}(3) - 1$$

The dependent variable $f(3) = 1$

5.1.5 Continuous and discontinuous functions

With a discontinuous function, the curve is always in two parts although in the case shown in **Figure 5.3**, we know the curve is actually continuous and meets in infinity.

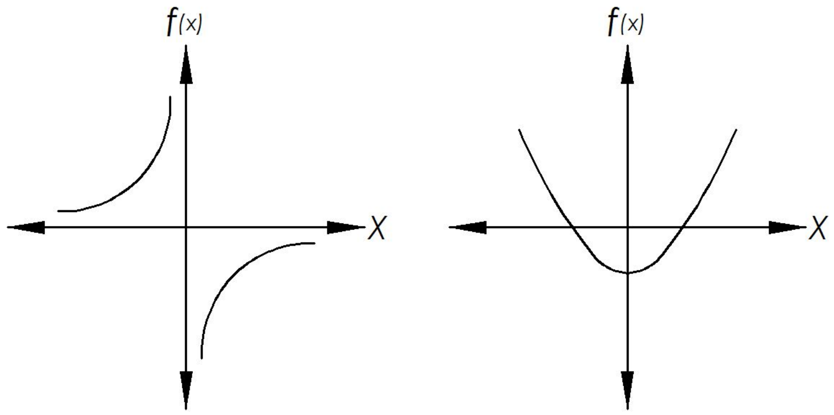


Figure 5.3 Discontinuous function Continuous function

A continuous function has a curve that is always in one part.

5.1.6 Functions and relations

Definition: Relation
A relation is simply a set of ordered pairs.

If we impose the following rule on a relation, it becomes a function.

Definition: Function
A function is a set of ordered pairs in which each x-element has only ONE y-element associated with it.

5.1.6.1 Inverse functions

A function has an inverse function if a line parallel to the X axes cuts the curve at one point only. See **Figure 5.4 (a).**

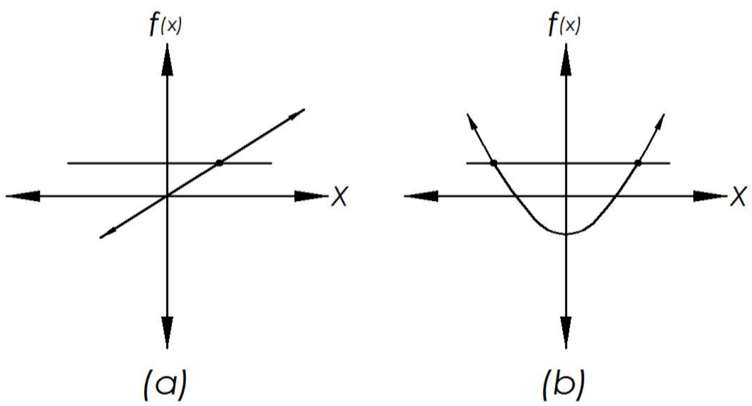


Figure 5.4 Curves representing functions

A function has not got an inverse function if a line parallel to the X axes cuts the curve at more than one point. See **Figure 5.4 (b).**

5.1.7 Symmetry

A curve can be symmetrical about a line or an axis. **Figure 5.5 (a)** shows symmetry about a line and **Figure 5.5 (b)** shows a curve symmetrical about an axes.

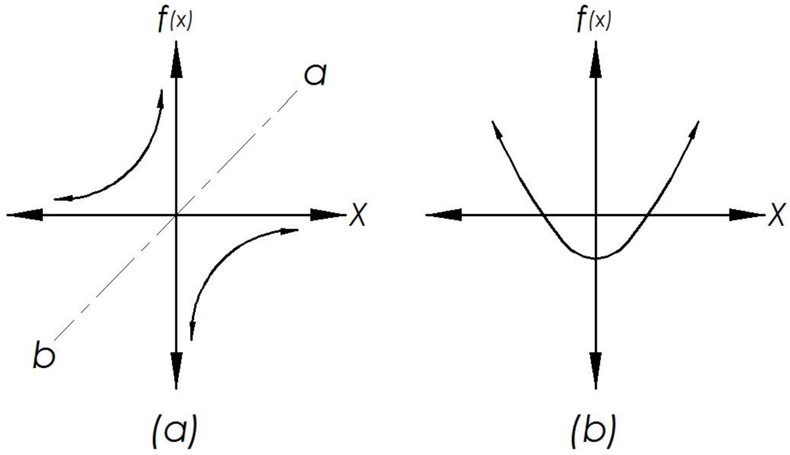


Figure 5.5 Curves with symmetry

5.2 The straight line graph

5.2.1 The function of a straight line curve

5.2.1.1 Gradient form

$$f(x) = m\,x + c$$

Where:
m is the gradient
c is the y intercept

Example: $f(x) = \frac{2}{3}x - 1$

5.2.1.2 General form

$$ax + by + c = 0$$

Convert the following equation that is in the general form to the gradient form:

$$2x + 5y = 5$$

$$5y = -2x + 5$$

$$y = -\frac{2}{5}x + 1$$

5.2.1.3 Intercept form
The line intercepts the x axes at a and the y axes at b:

$$\frac{x}{a}+\frac{y}{b}=1$$

Convert the following equation that is in the intercept form to the general form and then the gradient form:

$$\frac{x}{2}+\frac{y}{-1}=1$$

Multiply both sides with the LCM:

$$x-2y=2$$

$$\text{General form} \quad x-2y-2=0$$

$$-2y=-x+2$$

$$\text{Gradient form} \quad y=\tfrac{1}{2}x-1$$

5.2.1.4 If the gradient and one ordered pair are known
Use the formula:

$$y-y=m\,(x-x)$$

A line passes through a point (-3;3) and the gradient is -1/2. Find the function in gradient form:

$$y-y=m\,(x-x)$$

$$y-3=-\tfrac{1}{2}\left(x-(-3)\right)$$

$$y-3=-\tfrac{1}{2}\,x-\tfrac{3}{2}$$

$$y=-\tfrac{1}{2}\,x-\tfrac{3}{2}+3$$

$$y=-\tfrac{1}{2}\,x+\tfrac{3}{2}$$

5.2.1.5 The two-point form
Use the formula:

$$\frac{y-y_1}{x-x_1}=\frac{y_2-y_1}{x_2-x_1}$$

When the line goes through two co-ordinates. The two co-ordinates are substituted into this formula.

Find the equation for a straight line that passes through the points:
(-1;-2) and (1;2):

$$\frac{y-y_1}{x-x_1}=\frac{y_2-y_1}{x_2-x_1}$$

$$\frac{y-(-2)}{x-(-1)}=\frac{2-(-2)}{1-(-1)}$$

$$\frac{y+2}{x+1}=\frac{4}{2}$$

$$4+2y=4+4x$$

$$4x-2y=0$$

$$y=\frac{2}{4}x=\frac{1}{2}x$$

5.2.2 Parallel lines and perpendicular lines

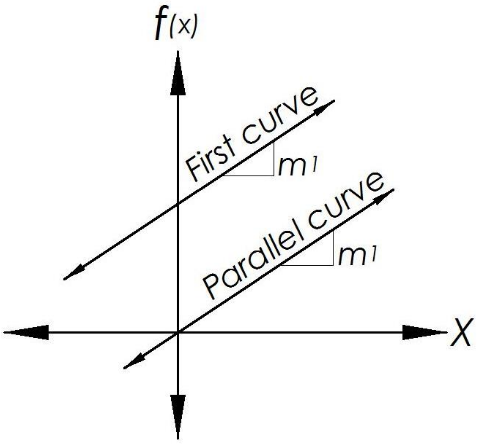


Figure 5.6 Parallel lines

Worked Example 5.1

Find the equation of a line (line 1) that goes through a point (0;1) and is parallel to $y=x+2$ (line 2).

Solution:

Gradient of (line 1) =1 (same as line 2)

$$m\,(x-x_1)=\;y-y_1$$

$$1\,(x - 0) = y - 1$$

$$y = x + 1$$

With perpendicular lines, it is obvious that m_1 and m_2 are different.

Use the formula:

$$m_1 \times m_2 = -1$$

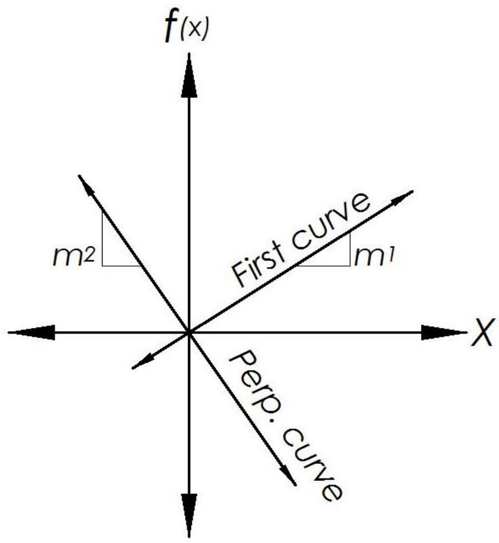


Figure 5.7 Perpendicular lines

Worked Example 5.2

Find the equation of a line (line 1) that passes through a point (2;2) and is perpendicular to (line 2) $y = 2x + 1$.

Solution:

Find the gradient on (line 1):

$$m_1 \times m_2 = -1$$

$$m_1 \times \frac{2}{1} = -1$$

$$m_1 = -\frac{1}{2}$$

$$-\frac{1}{2}\,(x - 2) = y - 2$$

$$y = -\frac{1}{2}x - 3$$

The angle of incline:

$$m = \tan\theta = \frac{y_2 - y_1}{x_2 - x_1}$$

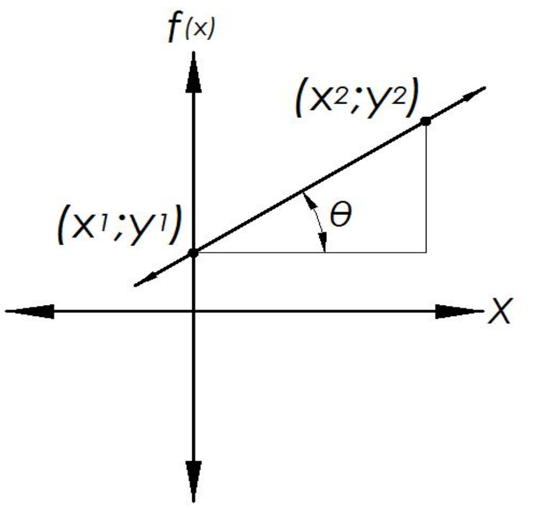


Figure 5.8 The angle of incline

5.2.3 Distance between two points

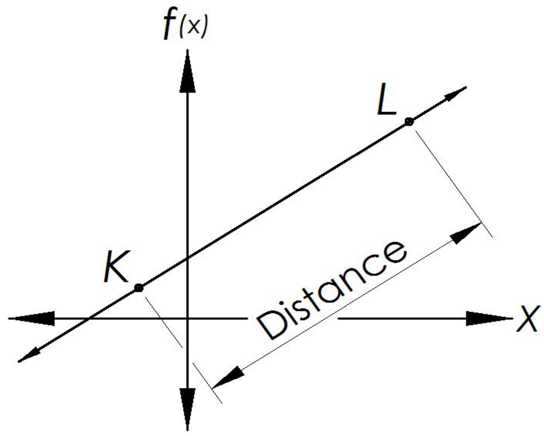


Figure 5.9 Distance between two points

Pythagoras:

$$\text{Length between K and L} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

5.3 The circle

The syllabus calls for circles with their centre at the point of origin. (0;0). The radius of the circle can be found in three ways:

5.3.1 The square method

When the radius is a perfect square.

Worked Example 5.3

Draw the circle with the equation:

$$x^2 + y^2 = 9$$

Solution:

The radius is 3 … the square root of 9.

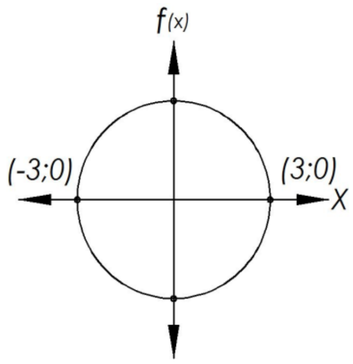


Figure 5.10 $x^2 + y^2 = 9$

5.3.2 The co-ordinate method

Worked Example 5.4

Draw the circle with the equation:

$$x^2 + y^2 = 17$$

Solution:

Try to write the square root of 17 another way:

$$\sqrt{17} = \sqrt{16} + \sqrt{1}$$

$$\text{From } \sqrt{16} \text{ and } \sqrt{1} \ = \ 4 \text{ and } 1$$

$$\text{The circle will run through the point } = (4; 1)$$

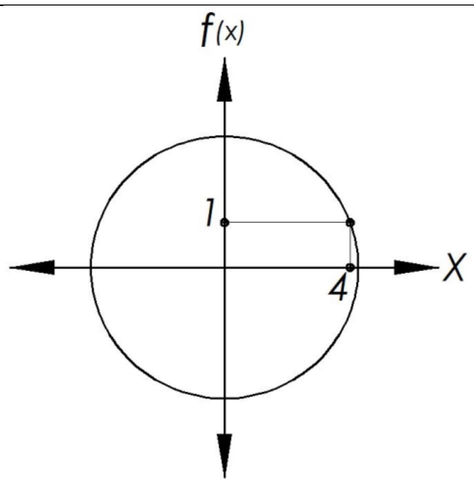


Figure 5.11 $x^2 + y^2 = 17$

5.3.3 The geometry method

Worked Example 5.5

Draw the circle with the equation:

$$x^2 + y^2 = 8$$

There are 3 steps to find this graph:

1. Find two parts of 8. ie: $2 \times 4 = 8$
2. On the x-axes, plot the two values namely 2 and 4. Then find the mid-point between the 2 and 4 on the x-axes. See **Figure 5.12**.

   Draw a construction circle with the centre on the x-axes and the mid-point line.

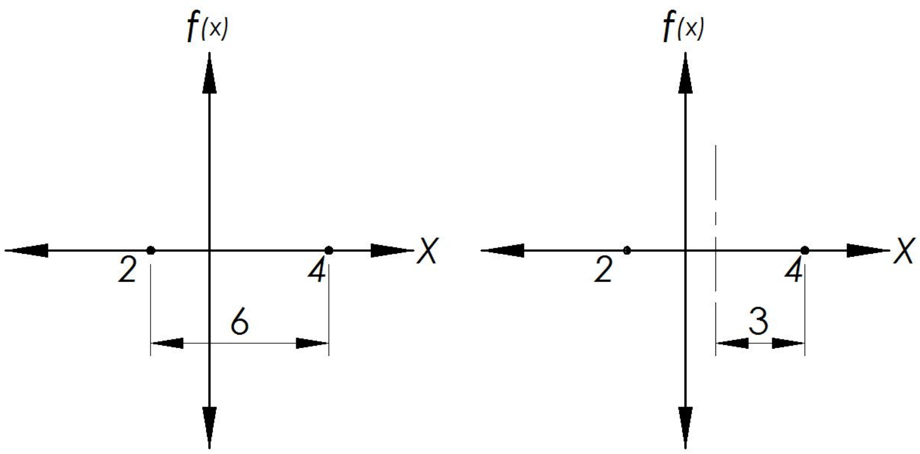


Figure 5.12 Step 2

3. Point A, where the construction circle crosses the y-axes, marks the point

where the circle graph intercepts the y-axes. The radius is as shown in **Figure 5.13**.

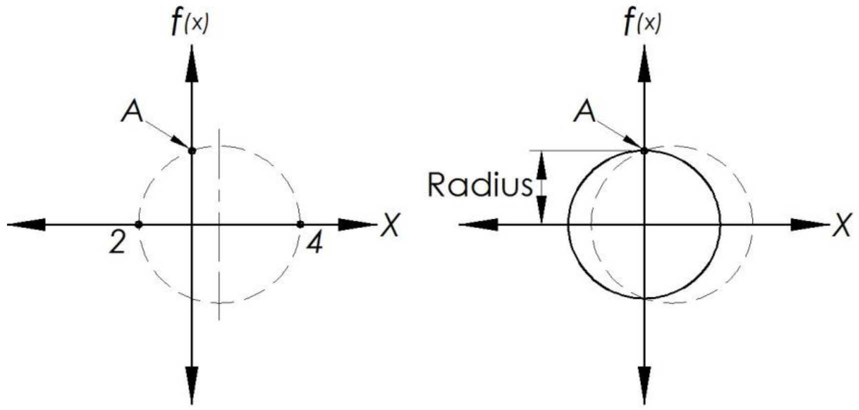


Figure 5.13 Step 3

5.3.4 Half circles

A negative function means that the half circle is drawn under the x-axes and a positive function will mean that the half circle will be on the positive half of the y-axes.

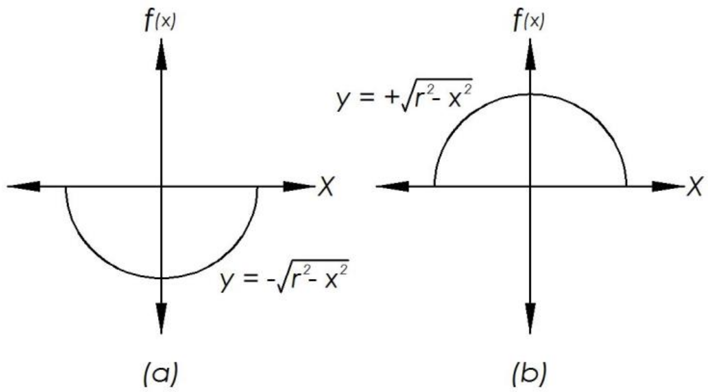


Figure 5.14 Half circle curves

5.4 The ellipse

Worked Example 5.6

Sketch the ellipse:

$$x^2 + 4y^2 = 16$$

Solution:

$$x^2 + 4y^2 = 16$$

Write in the standard form:

$$\frac{x^2}{16} + \frac{4y^2}{16} = 1$$

$$\frac{x^2}{16} + \frac{y^2}{4} = 1$$

$$\frac{x^2}{4^2} + \frac{y^2}{2^2} = 1$$

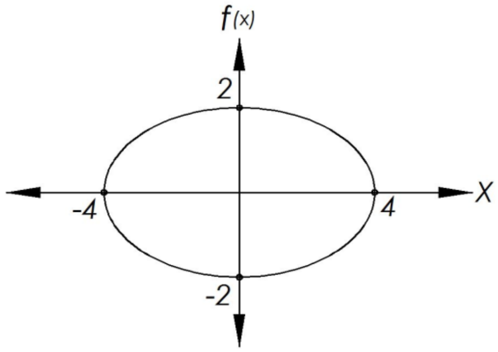


Figure 5.15 Ellipse

5.5 Hyperbolas

The equation $xy = k$

Worked Example 5.7

Sketch the hyperbola:

$$x\,y = 10$$

Solution:

Create a table for the positive half with minimum 4 columns:

x	1	2	2	1
y	10	5	5	10

Table 5.1

Do the same for the negative half:

x	-1	-2	-2	-1
y	-10	-5	-5	-10

Table 5.2

Plot the graph:

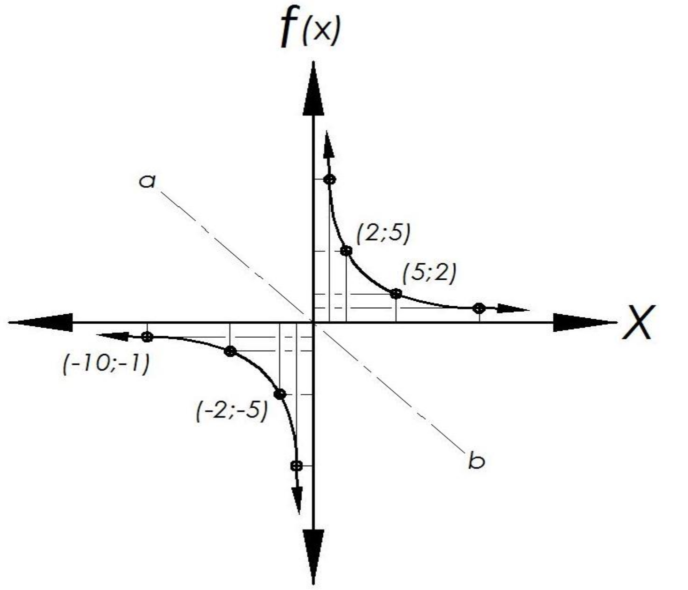


Figure 5.16 Hyperbola

5.6 The parabola

General equation:

$$y = ax^2 + bx + c$$

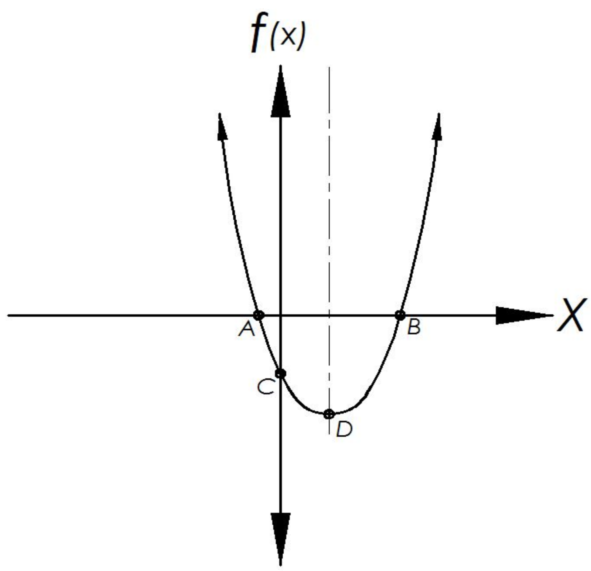


Figure 5.17 Parabola

Four points need to be calculated before the curve can be drawn. They are shown in **Figure 5.17**.

Worked Example 5.8

Sketch the graph:

$$y = x^2 + 2x - 3$$

Solution:

1. Find the x- intercepts

$$0 = x^2 + 2x - 3$$

$$0 = (x + 3)\ (x - 1)$$

$$(x + 3) = 0 \quad or \quad (x - 1) = 0$$

$$x = -3 \quad or \quad x = 1$$

2. find the axis of symmetry

$$D = \frac{A + B}{2} = \frac{-3 + 1}{2} = -1$$

3. Turning point

Substitute D into the equation:

$$y = (-1)^2 + 2(-1) - 3 = -4$$

4. The y-intercept

Substitute $x = 0$ into the equation:

$$y = (0)^2 + 2(0) - 3 = -3$$

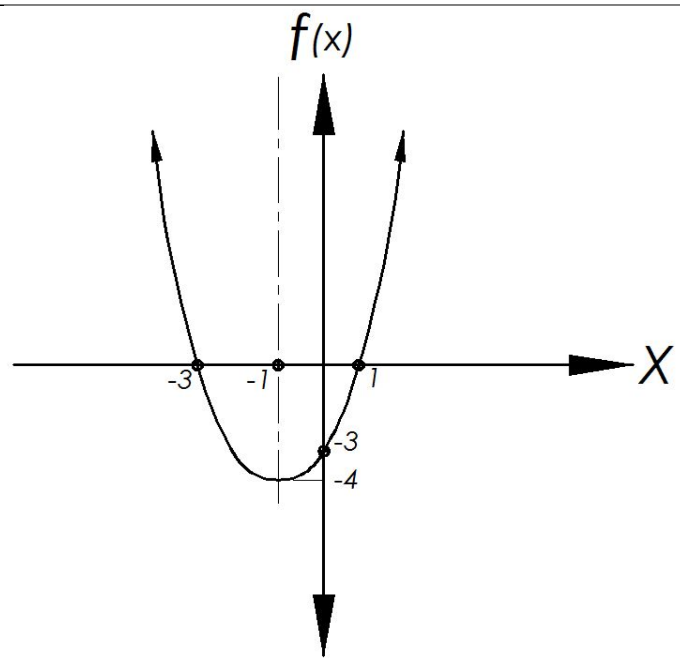


Figure 5.18 Parabola

5.7 The curve of the logarithmic function

5.7.1 The exponential function

The general equation $y = a^x$

Worked Example 5.9

Sketch the curve:

$$y = 2^x$$

Solution:

Create a table with minimum 5 columns:

$$y = 2^x$$

x	-2	-1	0	1	2
y	0.25	0.5	1	2	4

Table 5.3

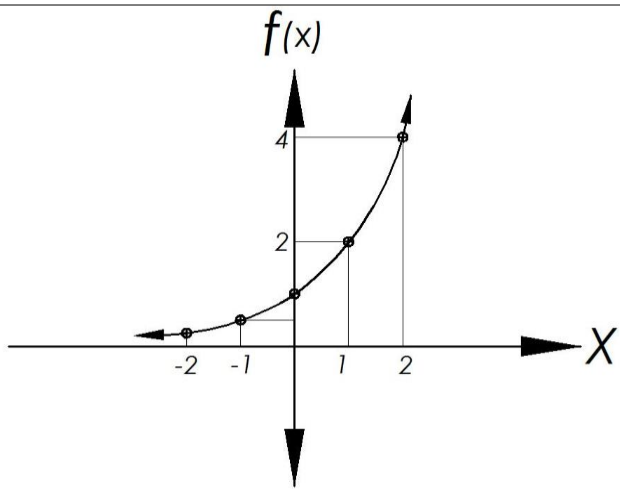


Figure 5.19 Exponential curve

5.7.2 The logarithmic function

The general equation $y = \log_a x$

Worked Example 5.10

Sketch the curve:

$$y = \log_2 x$$

Solution:

First write in exponential form:

$$y = \log_2 x \quad \text{Becomes} \quad 2^y = x$$

Create a table with minimum 5 columns:

$$y = 2^x$$

y	-2	-1	0	1	2
x	0.25	0.5	1	2	4

Table 5.4

Plot the y-values and then the corresponding x-values:

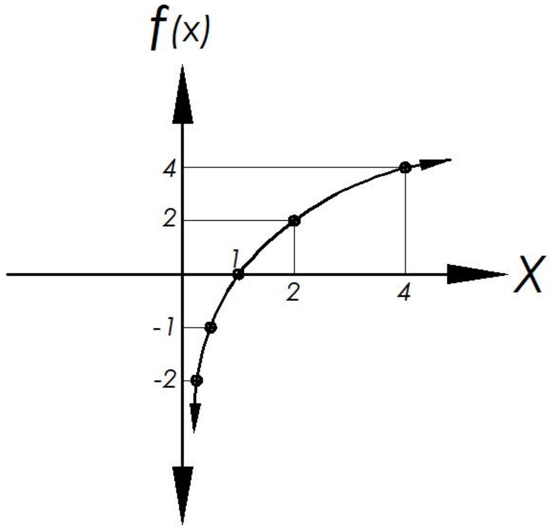


Figure 5.20 Logarithmic curve

5.8 The curve trigonometric functions

$$\text{The basic equation } y = \sin x \text{ between } 0° \; and \; 360°$$

Worked Example 5.11

Sketch the curve:

$$y = \sin x \text{ between } 0° \; and \; 360°$$

Solution:

Create a table with enough columns to obtain a smooth curve:

x	0	30	90	180	270	360
$y = \sin x$	0	0.5	1	0	-1	0

Table 5.5

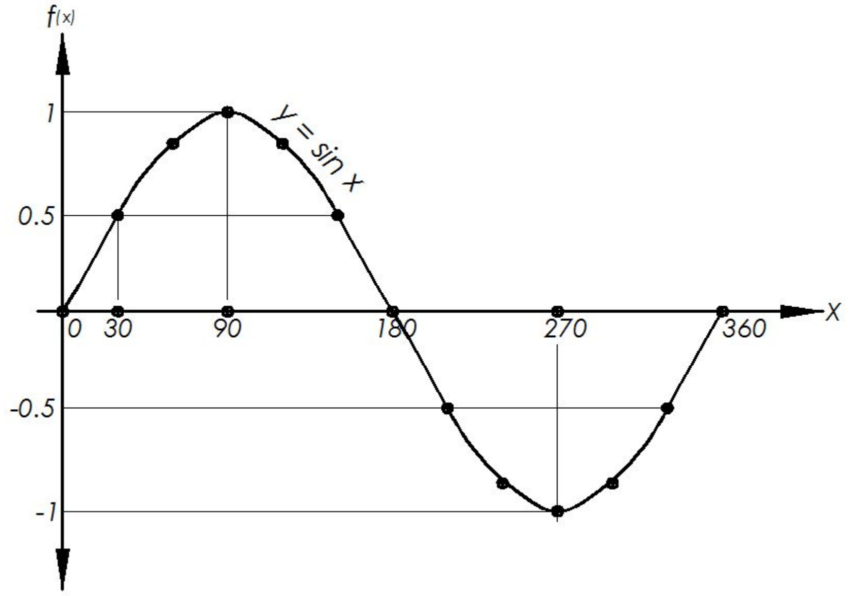


Figure 5.21 Trigonometric curve

5.9 Amplitude, frequency and phase shift

Changing the amplitude of a wave increases or decreases its height. The frequency is the number of waves in a set time. The phase shift moves the entire curve along the x-axis.

$y = 2\sin x$ the 2 before the sin x influences the amplitude

$y = \sin 2x$ the 2 in this position influences the freqency

$y = \sin(x + 10°)$ the 10° in this position changes the phase

Worked Example 5.12

Sketch the curves:

$$y = 2\sin x\ ;\ \sin 2x \ and \ \sin(x + 10°) \quad \text{between} \quad 0° \ and \ 360°$$

Solution:

$y = 2\sin x$

x	0	30	90	180	270	360
$y = 2\sin x$	0	1	2	0	-2	0

Table 5.6

$y = \sin 2\,x$

x	0	30	90	180	270	360
$y = \sin 2x$	0	0.866	0	0	0	0

Table 5.7

$y = \sin(x + 10°)$

x	0	30	90	180	270	360
$y = \sin 2x$	0.174	0.643	0.985	-0.174	-0.985	0.174

Table 5.8

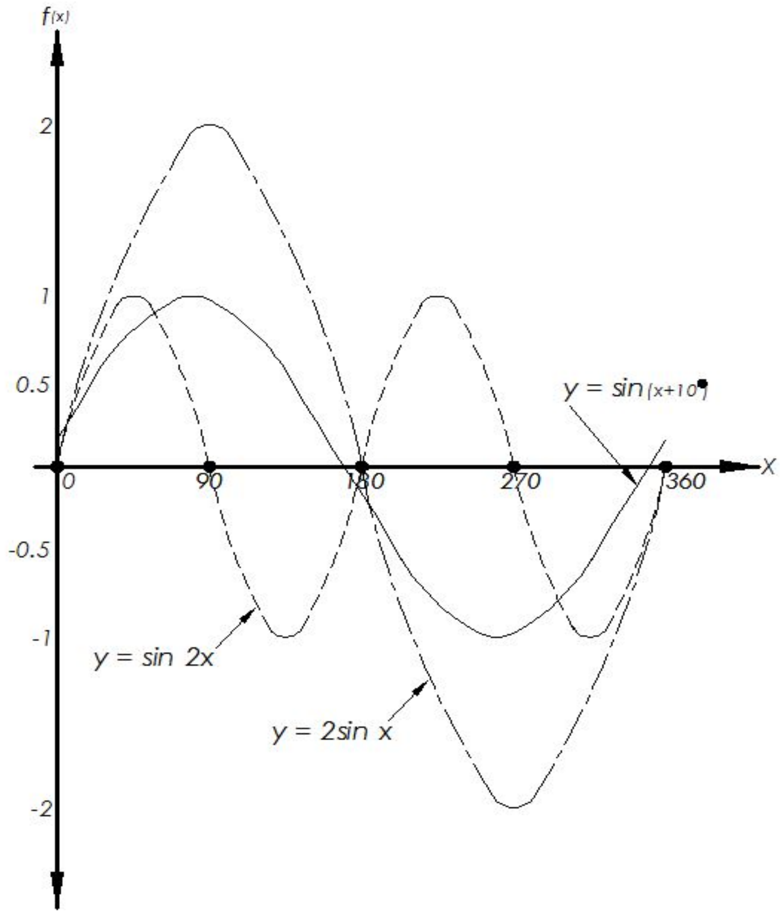


Figure 5.22 Trigonometric curves

In **Figure 5.22** notice how the phase shift of 10° causes the curve to shift on the x-axis and the change in amplitude causes an increase in height and the frequency change increases the number of waves.

Worked Example 5.13

Sketch the graph of $y = \sec x\ ; 0° \leq x \leq 360°$.

Solution:

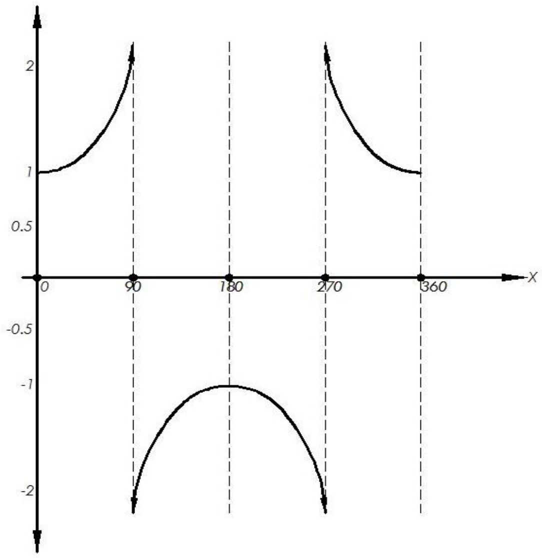


Figure 5.23

Worked Example 5.14

Sketch the graph of $y = \tan x\ ; 0° \leq x \leq 360°$.

Solution:

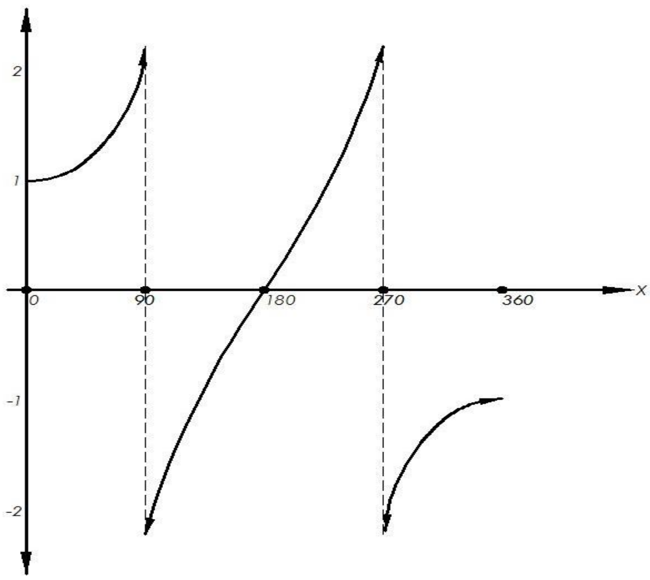


Figure 5.24

Worked Example 5.15

Sketch the graph of $y = \cos ecx\ ; -\pi \le x \le \pi$.

Solution:

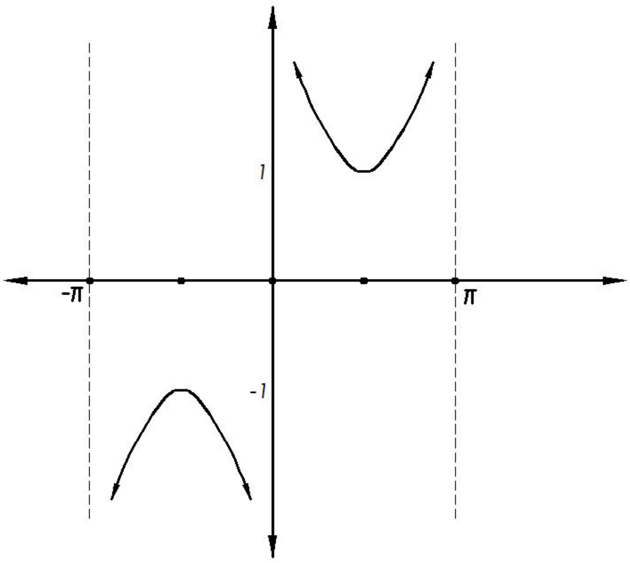


Table 5.25

5.10 The curve of cubic functions

$$\text{The equation } y = \ x^3 + \ x^2 + x + C$$

Worked Example 5.16

Sketch the curve of:

$$y = \ x^3 - 6x$$

Solution:

Find the turning point:

Use differentiation:

$$f(x) = \ x^3 - 6x$$

$$f'(x) = \ 3x^2 - 6$$

When $f'(x) = \ 0$ then solve for x in the normal way:

$$3x^2 - 6 = 0$$

$$x^2 = \frac{6}{3}$$

$$x = \pm 2 = +2 \text{ and } -2$$

The y-value at the turning point:

$$f(x) = x^3 - 6x$$

Substitute $x = +2$ $\quad f(2) = 2^3 - 12 = -4$

Substitute $x = -2$ $\quad f(-2) = (-2)^3 + 12 = +4$

$$\text{Turning points} = \left(\frac{x\,;\quad y}{-2\,;\ 4}\right)\left(\frac{x\,;\ y}{2\,;\ -4}\right)$$

The roots:

$$f(x) = x^3 - 6x$$

$$x^3 - 6x = 0$$

$$x\,(x^2 - 6) = 0$$

$$x = 0 \quad \text{and} \quad (x^2 - 6) = 0$$

$$x = 0 \qquad x = +2.45 \qquad x = -2.45$$

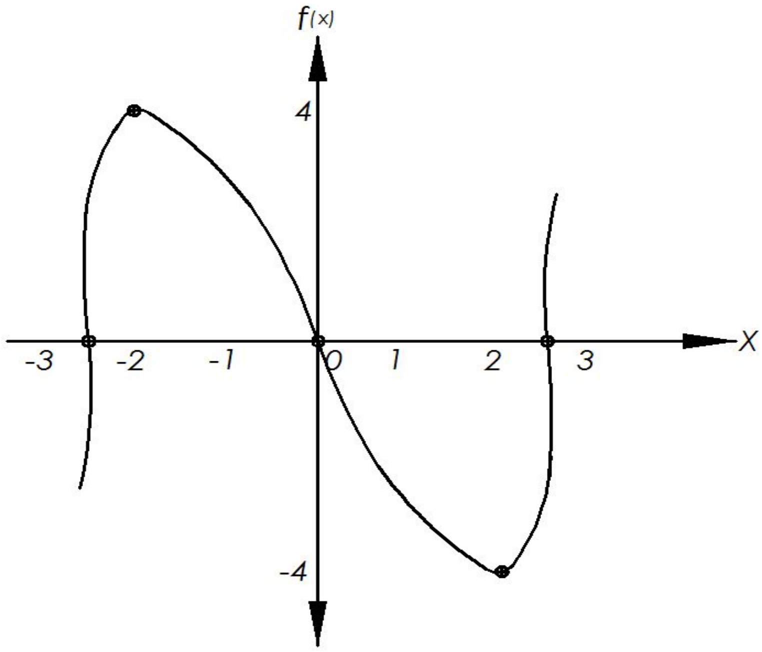


Figure 5.26

Activity 5.1

1. Find the slope and Y-intercept:
   $2(y-2)=3(x-1)$

2. Find the equation of the line:
   with gradient -2 and which intercepts the y-axis at 1;

3. Find the equation of the line:
   through the point (1;2) and parallel to $y+2=3x$;

4. Find the equation of the line:
   through the point (-3;2) and perpendicular to $y-2x-4=0$;

5. Find the equation of the straight line that runs through the point (-1;1) and that is perpendicular to the line that connects the points (4;0) and (0;1).

Answers:

1. $m=\frac{3}{2}; c=\frac{1}{2}$
2. $y=-2x+1$
3. $y=3x-1$
4. $2y+x-1=0$
5. $y=4x+3$

Activity 5.2

1. Sketch the graph of $\frac{x^2}{36}-\frac{y^2}{49}=1$
2. What is the domain of the graph of $\frac{x^2}{36}-\frac{y^2}{49}=1$
3. Sketch the graph of $y=x^2+5$
4. Is the graph of $y=x^2+5$ symmetric about the y-axis?
5. Sketch the graph of $x^2+y^2=12{,}25$
6. Sketch the graph of $y=-x^2+2x+3$
7. Is the graph of $y=-x^2+2x+3$ discontinuous or not discontinuous?

8. What is the range of the graph of $y = -x^2 + 2x + 3$?
9. Sketch the graph of $xy = -3$
10. Sketch the graph of $y = 6 + 4x - 2x^2$
11. Is the graph of $y = 6 + 4x - 2x^2$ continuous or discontinuous?
12. Sketch the graph of $y = \ell nx$
13. Sketch the graph of $y = -x^2 + 5x - 4$
14. Is the graph of $y = -x^2 + 5x - 4$ in QUESTION 13 continuous or discontinuous?
15. Sketch the graph of $xy = -1$.
16. Sketch the graph of $\frac{x^2}{9} + \frac{y^2}{4} = 1$.
17. Is the graph of $\frac{x^2}{9} + \frac{y^2}{4} = 1$ symmetrical or asymmetrical about the x-axis?

Answers:

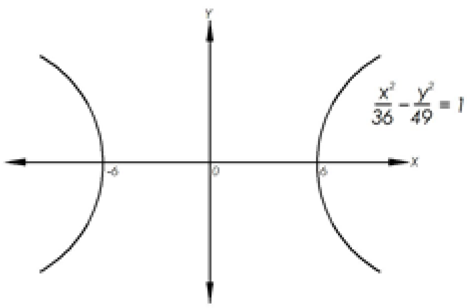


Figure 5.27

2. Domain: $\{x: -6 \leq x \leq 6; x \in \mathbb{R}\}$

   Definition: $\{x: -6 \leq x \leq 6; x \in \mathbb{R}\}$

3.

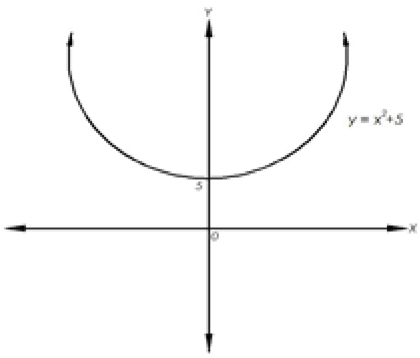


Figure 5.28

4. Yes, it is symmetric about the y-axis.
5.

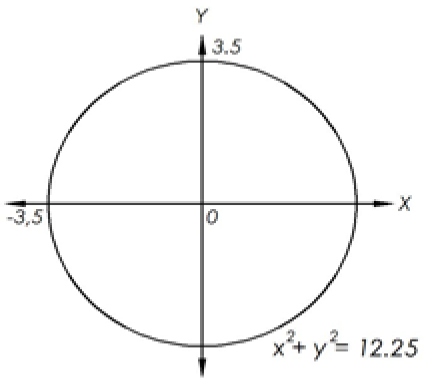


Figure 5.29

6.

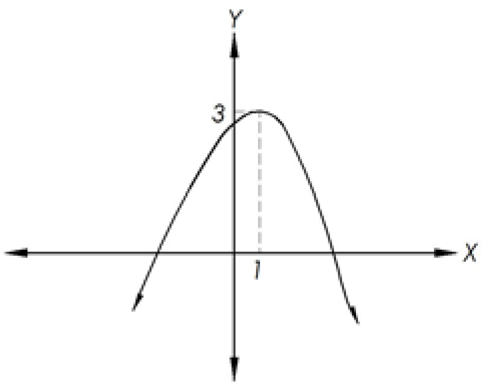


Figure 5.30

7. Continuous
8. $[4; -8]$
9.

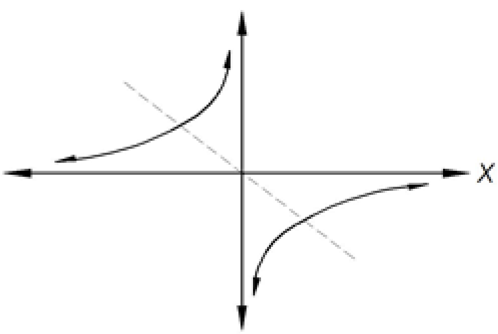


Figure 5.31

10.

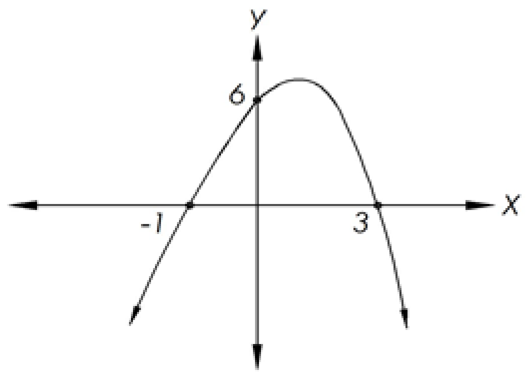


Figure 5.32

11. Continuous
12.

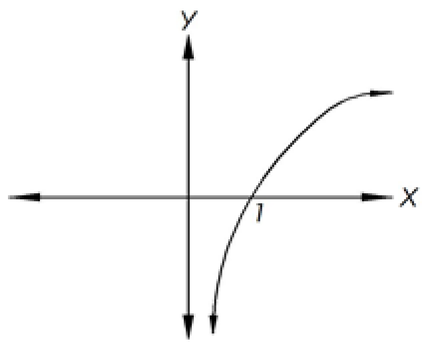


Figure 5.33

13.

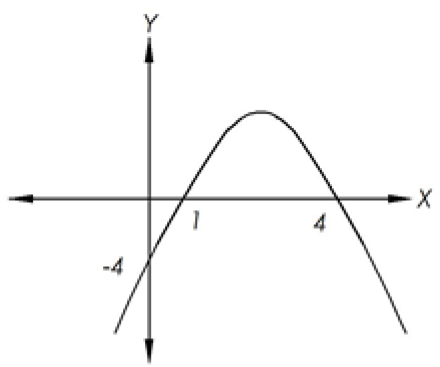


Figure 5.34

14. Continuous
15.

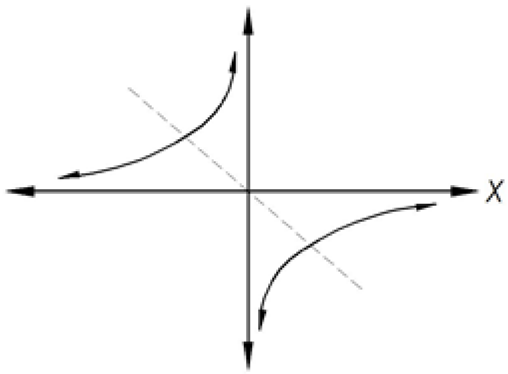


Figure 5.35

16.

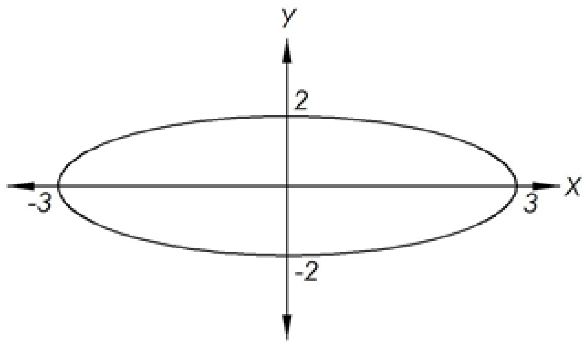


Figure 5.36

17. Symmetrical

18. No

	Self-Check	
I am able to:	**Yes**	**No**
• Identify domain and range, dependent and independent variables, functions and relations, points of symmetry, continuous and discontinuous functions, inverse functions and relations		
• Describe and draw graphs without tables of values or point for point plotting		
If you have answered 'no' to any of the outcomes listed above, then speak to your facilitator for guidance and further development.		

Module 6

Limits and Differentials

Learning Outcomes
On the completion of this module the student must be able to:

- Describe the binomial theorem in general terms
- Expand a binomial to four terms
- Calculate simple limits with the theorems on limits
- Define differentiation and derive the expression for the rate of change
- Differentiate to standard forms
- Calculate by applying the chain rule to a function of a function
- Determine the first derivatives of polynomials
- Apply the product and quotient rules for differentiation
- Determine the second derivatives of trigonometric functions

6.1 Introduction

The essence of calculus is the derivative. The derivative is the instantaneous rate of change of a function with respect to one of its variables. This is equivalent to finding he slope of the tangent line to the function at a point.

6.2 The binomial theorem

Take a binomial like $(x + y)$ and square it.

$$(x + y)^2$$

Now expand the binomial:

$$(x + y)^2 = x^2 + 2xy + y^2$$

Notice a few things about the answer:

- The coefficient's have the pattern: 1, 2, 1
- n, the index for the x is 2 for the first term, 1 for the second term and 0 for the third term
- n, the index for the y is 0 for the first term, 1 for the second term and 2 for the third term
- The binomial is squared so the answer has three terms

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

Figure 6.1 Pascal's triangle

6.2.1 Pascal's triangle

The binomial coefficients appear as the entries of Pascal's triangle where each entry is the sum of the two above it.

Note that in the example above, the coefficient's 1, 2, 1 appear in the second row of the triangle.

Study the expansions below and relate them to Pascal's triangle:

$(a+b)^0 = 1$
$(a+b)^1 = a + b$
$(a+b)^2 = a^2 + 2ab + b^2$
$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

6.2.2 Negative integer or a fractional integer

What occurs if "n" is a negative integer or a fraction? Sir Isaac Newton used long division to expand $(a+b)^n, n < 0$. The expansion, however, yields an infinite number of terms. For example, find the expansion of $(a+b)^{-2}$ by division:

Did you know?

Figure 6.2

Isaac Newton (1642-1727)
Nationality: English
Famous For:*Mathematical Principles of Natural Philosophy*

The book of Sir Isaac Newton, *Mathematical Principles of Natural Philosophy*, became the catalyst to understanding mechanics. He is also the person credited for the development of the binomial theorem.

Worked Example 6.1

Expand $(a+2b)^4$ by means of the Binomial Formula:

Solution:

$$(a+2b)^4 = a^4 + 4a^3(2b) + \frac{4.3.a^2(2b)^2}{2!} + \frac{4.3.2.a(2b)^3}{3!} + \frac{4.3.2.1.a^0(2b)^4}{4!} + 0$$
$$= a^4 + 8a^3b + 24a^2b^2 + 32ab^3 + 16b^4 \text{ (no restrictions)}$$

Worked Example 6.2

Determine the first three terms in the expansion of $(1-x)^{-1}$.

Solution:

$$(1-x)^{-1} = [1+(-x)]^{-1} = 1 + (-1)(1)(-x) + \frac{(-1)(2)(1)(-x)^2}{2!} + \cdots$$
$$= 1 + x + x^2 + \cdots if\ |x| < 1$$

Worked Example 6.3

Expand the binomial:

$$(a+b)^{-2}$$

Solution:

$$(a+b)^{-2} = \frac{1}{(a+b)^2} = \frac{a}{a^2+2ab+b^2}$$

Now divide $a^2+2ab+b^2$ into 1:

$$\frac{1}{a^2} - \frac{2b}{a^3} + \frac{3b^2}{a^4} - \frac{4b^3}{a^5} + \cdots$$

$$2a^2 + 2ab + b^2 \;\Big|\; 1 + 0 + 0$$

$$1 + \frac{2b}{a} + \frac{b^2}{a^2}$$

$$0 - \frac{2b}{a} - \frac{b^2}{a^2}$$

$$-\frac{2b}{a} - \frac{4b^2}{a^2} - \frac{2b^3}{a^3}$$

$$0 + \frac{3b^2}{a^2} + \frac{2b^3}{a^3}$$

$$\frac{3b^2}{a^2} + \frac{6b^3}{a^3} + \frac{3b^4}{a^4}$$

$$0 + \frac{4b^3}{a^3} + \frac{3b^4}{a^4}$$

$$\vdots$$

$$(a+b)^{-2} = \frac{1}{a^2} - \frac{2b}{a^3} + \frac{3b^2}{a^4} - \frac{4b^3}{a^5} + \cdots$$

Note:
We are expected to expand binomials up to four terms only.

6.3 Limits

Before we go into limits, it's important to understand functional notation and the concept of the increment.

6.3.1 Functional notation

The term 'function' was first used by Leibniz in 1673 to indicate the dependence of one variable on another.

Worked Example 6.4

Given that $f(x)$ is a function of x

And the equation:

$y = f(x) = x^2 - 2x + 1,$

Find the values of: $f(1)$, $f(-1)$ and $f(x+h)$

Solution:

$$f(1) = (1)^2 - 2(1) + 1 = 0$$

$$f(-1) = (-1)^2 - 2(-1) + 1 = 4$$

$$y(x+h) = (x+h)^2 - 2(x+h) + 1 = x^2 + 2xh + h^2 - 2x - 2h + 1$$

Did you know?

Figure 6.3

Gottfried Wilhelm Leibniz (1646-1716)
Nationality: German
Famous For: *Infinitesimal calculus*

The work of Leibniz on infinitesimal calculus was one completely separate from Isaac Newton. His mathematical notation continues to be in use. He also proposed the mathematical principle known as the *Transcendental Law of Homogeneity*. His refining of the binary system has become foundational in mathematics.

6.3.2 Increments

When very small values of variables are used, they are known as increments.
For example, if x changes by a small amount, this small amount is known as Δx.

Δx means an increment in x.

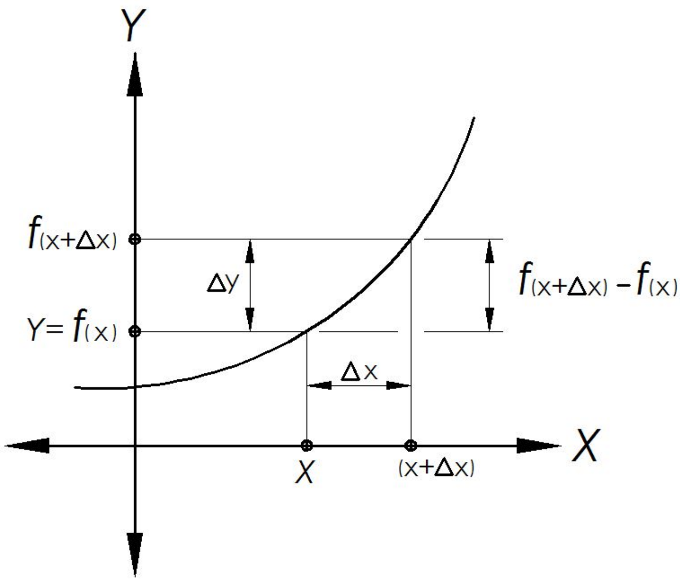
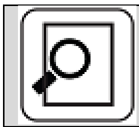


Figure 6.4 The increase of x and y

6.3.3 The concept of the limit

As the independent variable x approaches a value, say 3, the dependent variable $f(x)$ or as it is known, the functional value will automatically approach a limit.



Worked Example 6.5

Find the limit of the function $f(x) = 2x - 5$ when x tends toward 3.

Solution:

To clearly show how this works, construct a table. (This is not done in practice.)

$$f(x) = 2x - 5$$

x	2.9	2.99	2.999	2.9999	3	3.0001	3.001	3.01	3.1
$f(x)$	0.8	0.98	0.998	0.9998	1	1.0002	1.002	1.02	1.2

Table 6.1

The table shows that when x approaches 3, $f(x)$ approaches 1. So the limit of the function $f(x) = 2x - 5$ when x tends towards 3 is 1.

It is written this way:

$$\lim_{x \to 3}(2x - 5) = 1$$

Worked Example 6.6

Find, without the use of a table:

$$\lim_{x\to 1}\left(\frac{x^2-3x+2}{x-1}\right)$$

Solution:

Remember x does not equal 1, it tends toward 1.

So we factorise:

$$\lim_{x\to 1}\frac{x^2-3x+2}{x-1}$$

$$= \lim_{x\to 1}\frac{(x-1)(x-2)}{x-1}$$

$$= \lim_{x\to 1}(x-2)$$

= -1

6.4 The derivative

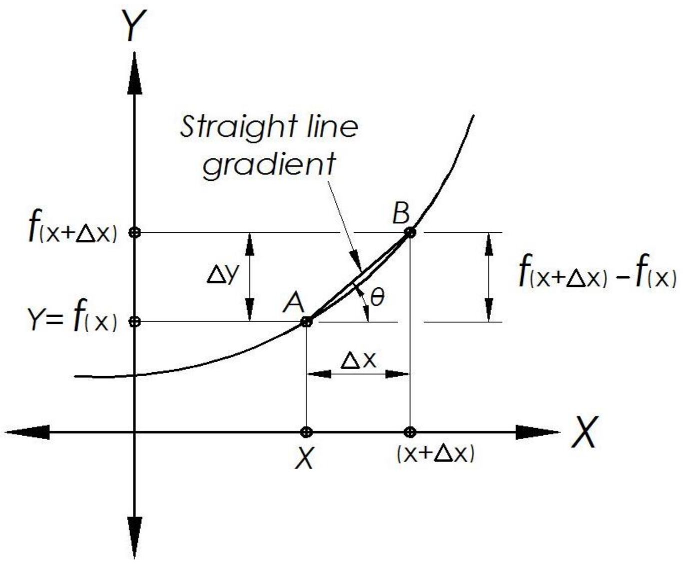


Figure 6.5 Finding the gradient of a curve

Figure 6.5 shows an attempt at finding the gradient of a curve between points A and B on the curve. Note that the distance between A and B is governed by Δx. This is not accurate.

If Δx was much smaller, then the distance between A and B would be smaller and the straight line gradient would be a little more accurate.

But if Δx tends toward 0. So small that people say it is infinitesimally small. Then the straight line gradient between A and B would be accurate.

At this level of smallness, we can say that the straight line gradient is a tangent to the curve.

From **Figure 6.5:**

$$\tan\theta = \frac{f(x+\Delta x)-f(x)}{(x+\Delta x)-x}$$

$$\tan\theta = \frac{f(x+\Delta x)-f(x)}{\Delta x}$$

$$\text{The gradient of the tangent} = \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

The derivative of $f(x)$ with respect to x is:

$$f'(x) = \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

6.4.1 Differentiation from first principles

Worked Example 6.7

Find $\frac{dy}{dx}$ or $f'(x)$ of $f(x) = x^2$

Solution:

Step1. Write down the function:

$$f(x) = x^2$$

Step 2. Find $(x+\Delta x)$:

$$f(x+\Delta x) = (x+\Delta x)^2$$

$$= x^2 + 2x.\Delta x + (\Delta x)^2$$

Step 3. Find $f(x + \Delta x) - f(x)$ and simplify:

$$f(x + \Delta x) - f(x) = x^2 + 2x.\Delta x + (\Delta x)^2 - x^2$$

$$= 2x.\Delta x + (\Delta x)^2$$

$$= \Delta x\,(2x + \Delta x)$$

Step 4. Find $f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta x\,(2x + \Delta x)}{\Delta x}$$

$$f'(x) = \lim_{\Delta x \to 0} (2x + \Delta x)$$

$$= 2x$$

Worked Example 6.8

Using first principles, find the gradient of $y = 2x^2 - 4x + 1$ at $x = -1$ (that is, find the derivative of $y = 2x^2 - 4x + 1$ at $x = -1$.

Solution:

1. $y(x) = 2x^2 - 4x + 1$

2. $y(x + h) = 2(x + h)^2 - 4(x + h) + 1$

$= 2x^2 + 2xh + 2h^2 - 4x - 4h + 1$

3. $y(x + h) - y(h) = 2x^2 + 4xh + 2h^2 - 4x - 4h + 1 - (2x^2 - 4x + 1)$

$= 2x^2 + 4xh + 2h^2 - 4x - 4h + 1 - 2x^2 + 4x - 1$

$= 4xh + h^2 - 4h$

$= h(4x + h - 4)$

4. $\therefore y'(x) = \lim_{h \to 0} \frac{y(x+h)-y(x)}{h}$

$= \lim_{h \to 0} \frac{h(4x+h-4)}{h}$

$= \lim_{h \to 0} (4x + h - 4)$

$= 4x - 4$

6.4.2 Standard forms of differentiation

The following notations can be used for differentiation:

$$y' \ or \ f'(x) \ or \ \frac{dy}{dx} \ or \ D_x$$

The formula for quickly working out what we just proved from first principles is:

To differentiate x^n do the following $n.x^{n-1}$

Worked Example 6.9

To differentiate $f(x) = 3x^4$

Solution:

$$f(x) = 3x^4$$
$$f'(x) = 4 \times 3x^{4-1}$$
$$= 12x^3$$

The derivatives in **Table 6.1** below should be memorized:

	$y = f(x)$	$\frac{dy}{dx}$
1	x^n	$n.x^{n-1}$
2	e^x	e^x
3	a^x	$a^x \ln a$
4	$\ln x$	$\frac{1}{x}$
5	$\log_a x$	$\frac{1}{x \ln a}$

Table 6.2 Standard derivatives

6.4.3 Differentiation of trigonometric functions

Proving that $\frac{dy}{dx}\sin x = \cos x$ is not required for the exams but it is provided to enhance your understanding.

Remember, finding the derivation of $\sin x$ is simply finding the gradient of the sin-wave form.

Worked Example 6.10

Demonstrate that: $\frac{d}{dx}\sin x = \cos x$

Solution:

1. $f(x) = \sin x$
2. $f(x+h) = \sin(x+h)$
3. $f(x+h) - f(x) = \sin(x+h) - \sin x$

$= 2\cos\frac{1}{2}(2x+h)\sin\frac{1}{2}h$ [see ∯ 6.2 identity (16)]

$= 2\cos\left(x+\frac{h}{2}\right)\sin\frac{h}{2}$

4. $\therefore \frac{d}{dx}f(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$= \lim_{h\to 0}\frac{2\cos\left(x+\frac{h}{2}\right)\sin\frac{h}{2}}{h}$

$= \lim_{h\to 0}\frac{\cos\left(x+\frac{h}{2}\right)\sin\frac{h}{2}}{\frac{h}{2}}$

$= \lim_{h\to 0}\cos\left(x+\frac{h}{2}\right)$ x $\lim_{h\to 0}\frac{\sin\frac{h}{2}}{\frac{h}{2}}$

$= \cos x$ x $\lim_{h\to 0}\frac{\sin\frac{h}{2}}{\frac{h}{2}}$ (*)

$= \cos x$ x 1 (**)

$= \cos x$

(*) ………. Because the sine function is continuous at 0.

(**) ……… Because $\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$ and if $h \to 0$, then $\frac{h}{2} = \theta \to 0$.

The derivatives in **Table 6.2** below should be memorized:

	$y = f(x)$	$\frac{dy}{dx}$
1	$\sin x$	$\cos x$
2	$\cos x$	$-\sec x$
3	$\tan x$	$\sec^2 x$
4	$\cot x$	$-\operatorname{cosec}^2 x$
5	$\sec x$	$\sec x \,.\tan x$
6	$\csc x$	$-\operatorname{cosec} x\,.\cot x$

Table 6.3 Standard trigonometric derivatives

6.4.4 Sum and difference rules

The derivatives in **Table 6.3** below should be memorized. The letter "k" is any constant.

	$y = f(x)$	$\frac{dy}{dx}$
1	k	0
2	kx^n	knx^{n-1}
3	$kf(x)$	$kf'(x)$
4	$f(x) \pm g(x)$	$f'(x) \pm g'(x)$

Table 6.4 Sum and difference rules

6.4.5 Product and quotient rules

The derivatives in **Table 6.4** below should be memorized.

	$y = f(x)$	$\frac{dy}{dx}$
1	$f(x) \times g(x)$	$f'(x) \times g(x) + f(x) \times g'(x)$
2	$\frac{f(x)}{g(x)}$	$\frac{g(x) \times f'(x) - g'(x) \times f(x)}{[g(x)]^2}$

Table 6.5 Product and quotient rules

Worked Example 6.11

Assume that $\frac{d}{dx}\sin x = \cos x$ and $\frac{d}{dx}\cos x = \sin x$.

Show that:

$$\frac{d}{dx}\tan x = \sec^2 x$$

Solution:

$Let\ y = \tan x \qquad \therefore y = \frac{\sin x}{\cos x}$

$$\therefore y' = \frac{\cos x \cos x - (\sin x \sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x}$$

$$= \sec^2 x$$

Worked Example 6.12

Assume that $\frac{d}{dx}\sin x = \cos x$ and show that:

$$\frac{d}{dx}\operatorname{cosec} x = -cosec\, x \cot x$$

Solution:

$$Let\ y = \operatorname{cosec} x \quad \therefore y = \frac{1}{\sin x}$$

$$\therefore y' = \frac{\sin x \times 0 - \cos x - \times 1}{\sin^2 x}$$

$$= \frac{-\cos x}{\sin^2 x}$$

$$= \frac{-\cos x}{\sin x} \times \frac{1}{\sin x}$$

$$= -\cot x \times cosec\ x$$

6.4.6 The chain rule

The chain rule is an extension of the sum and difference rules and the product and quotient rules. The chain rule states:

$$\text{If } \ y = y\,[u\,(x)] \ \text{ then } \ \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Worked Example 6.13

The following functions are differentiated with respect to the appropriate variable and the chain rule:

1. $y = \tan x \Rightarrow \frac{dy}{dx} = \frac{d}{dx}\tan(4x)$
2. $y = (2x)^3 \Rightarrow \frac{dy}{dx} = 3(2x)^2 \times \frac{d}{dx}(2x)$
3. $y = tan^{x^2} \Rightarrow \frac{dy}{dx} = \frac{d}{dx}e^{\tan x^2}$
4. $y = \ell n 3x \Rightarrow \frac{dy}{dx} = \frac{1}{3x} \times \frac{d}{dx}(3x)$

Solution:

1. $y = \tan x \Rightarrow \frac{dy}{dx} = \frac{d}{dx}\tan(4x)$

$= \sec^2 4x \times \frac{d}{dx}(4x)$

$= \sec^2 4x \times 4$

$= 4\sec^2 4x$

2. $y = (2x)^3 \Rightarrow \frac{dy}{dx} = 3(2x)^2 \times \frac{d}{dx}(2x)$

$= 12x^2 \times 2$

$= 24x^2$

3. $y = tan^{x^2} \Rightarrow \frac{dy}{dx} = \frac{d}{dx}e^{\tan x^2}$

$= e^{\tan x^2} \times \frac{d}{dx}\tan x^2$

$= e^{\tan x^2} \times sec^2 x^2 \times \frac{d}{dx}x^2$

$= e^{\tan x^2} \times sec^2 x^2 \times 2x$

$= 2 \times sec^2 x^2 e^{\tan x^2}$

4. $y = \ell n 3x \Rightarrow \frac{dy}{dx} = \frac{1}{3x} \times \frac{d}{dx}(3x)$

$= \frac{1}{3x} \times 3$

$= \frac{1}{x}$

Worked Example 6.14

Differentiate the following with respect to x:

$$y = \frac{\cos 2x}{\cos x + \sin x} - \frac{6}{\sqrt{x}} - \frac{1}{10e^2} + \frac{4}{\cos x}$$

Solution:

$y = \frac{\cos 2x}{\cos x + \sin x} - \frac{6}{\sqrt{x}} - \frac{1}{10e^2} + \frac{4}{\cos x}$

$y = \cos x - \sin x - 6x^{-\frac{1}{2}} - \frac{1}{10}e^{-x} + 4\sec x$

$\frac{dy}{dx} = -\sin x - \cos x + 3x^{-\frac{1}{2}} + \frac{1}{10}e^{-x} + 4\sec x \tan x$

Worked Example 6.15

Given:

$y = x^3 - 6x^2 + 11x - 6$

Calculate, with the aid of differentiation, the co-ordinates of the minimum and the maximum turning points. Distinguish between the minimum and the maximum points by using the second derivative.

Solution:

$\frac{dy}{dx} = 3x^2 - 12x + 11 = 0$

$x = \frac{12 \pm \sqrt{144 - 4(3)(11)}}{6}$

$x_1 = 2{,}577$ and $1{,}423$

$y_1 = (2{,}577)^3 - 6(2{,}577)^2 + 11(2{,}577) - 6$

$y_1 = -0{,}385$

$y_2 = (1{,}423)^3 - 6(6{,}423)^2 + 11(1{,}423) - 6$

$y_2 = 0{,}385$

$TP = (2{,}577; -0{,}385)$ and $(1{,}423; 0{,}385)$

$\frac{d^2y}{dx^2} = 6x - 12$

$= 6(2{,}577) - 12$

$= 3{,}462$

$= +$ minimum

$= 6(1{,}423) - 12$

$= -3{,}462$

$=$ (maximum)

Worked Example 6.16

Given:

$y = \frac{x^3}{3} - \frac{x^2}{2} - 2x + 7$

Calculate, with the aid of differentiation, the co-ordinates of the maximum and the minimum turning points.

Solution:

$$\frac{dy}{dx} = x^2 - x - 2$$

$$x^2 - x - 2 = 0$$

$$(x-2)(x+1) = 0$$

$$x_1 = 2 \qquad x_2 = -1$$

$$y_1 = \frac{2^3}{3} - \frac{2^2}{2} - 2(2) + 7$$

$$= 4{,}667$$

$$y_2 = \frac{(4)^3}{3} - \frac{1}{2} - 2(1) + 7$$

$$= 4{,}8333$$

Activity 6.1

Expand the following powers:

1. $(p+q)^4$
2. $(r+s)^4$
3. $(2x+3y)^3$
4. $\left(2a+\frac{b}{3}\right)^3$

Answers:

1. $p^4 + 4p^3q + 6p^2q^2 + 4pq^3 + q^4$
2. $r^4 + 4r^3s - 6r^2s^2 + 4rs^3 + s^4$
3. $8x^3 + 36x^2y + 54xy^2 + 27y^3$
4. $8a^3 + 4a^2b + \frac{2ab^2}{3} + \frac{b^3}{27}$

Activity 6.2

Expand the following powers using the Binomial Theorem.

1. $(1+x)^4$
2. $(1-3x)^3$

3. $\left(1-\frac{x}{3}\right)^4$
4. $\left(\frac{1}{2}-x\right)^4$
5. $(2-3a)^4$
6. $(1+2x)^{-3}$
7. $(1+2x)^{\frac{1}{2}}$

Answers:

1. $1+4x+6x^2+4x^3+x^4$
2. $1-9x+27x^2-27x^3$
3. $1-\frac{4x}{3}+\frac{2x^2}{1}-\frac{4x^3}{27}+\frac{x^4}{81}$
4. $\frac{1}{16}-\frac{x}{2}+\frac{3x^2}{2}-2x^3+x^4$
5. $16-96a+216a^2-216a^3+18a^4$
6. $1-6x+24x^2-80x^3+\cdots;|x|<\frac{1}{2}$
7. $1+x-\frac{x^2}{2}+\frac{x^3}{2}+\cdots;|x|<\frac{1}{2}$

Activity 6.3

Determine the following limits:

1. $\lim_{x\to 2}(4x^2-5x)$
2. $\lim_{x\to 1}\frac{x^2-2x+1}{x-1}$
3. $\lim_{x\to \infty}\frac{(x+1)(2+x)}{2x^3+x}$
4. $\lim_{x\to 3}\frac{x^3-27}{x^2-9}$

Answers:

1. 6
2. 0
3. ½
4. $\frac{9}{2}$

Activity 6.4

1. Differentiate $y = x^3$ with respect to x by using he definition.
2. Find the derivative of $y = \frac{1}{x^2}$, with the aid of the definition.
3. Determine $y'(x)$ if $y = x + \frac{1}{x}$, using the definition.
4. Determine the derivative of $y = 6x^2 - 2x + \frac{3}{x}$, by using the definition.

Answers:

1. $3x^2$
2. $3.\frac{-2}{x^3}$
3. $1 - \frac{1}{x^2}$
4. $12x - 2 - \frac{3}{x^2}$

Activity 6.5

1. $y = (x^2 + 3)(2x^2 - 1)$
2. $y = x^{3/2}(x^2 - 3)$
3. $y = \frac{\sqrt{x}}{x^2 - x}$
4. $y = \frac{x+b}{x-b}$

Answers:

1. $(1)2x(4x^2+5)$
2. $\frac{1}{2}\sqrt{2}(7x^2-9)$
3. $\frac{\sqrt{x}(1-3x)}{2(x^2-x)^2}$
4. $\frac{-2b}{(x-b)^2}$

Activity 6.6

1. $y = \sin 3x$
2. $y = \ tan^2 2x^2$
3. $y = (1-3x^2)^4$
4. $y = \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^3$

Answers:

1. $3\cos 3x$
2. $12x\tan^2 2x^2 \sec^2 2x^2$
3. $-24x(1-3x^2)^3$
4. $\frac{3}{2}\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^2\left(\frac{1}{\sqrt{x}}-\frac{1}{x\sqrt{x}}\right)$

Activity 6.7

1. Differentiate from the first principles if $y = x^2 - 3x$.
2. Differentiate with the aid of Quotient rule if:

   $$y = \frac{\sin x}{\sec x}$$

3. Given: $y = \frac{x^5}{2^x}$

   Differentiate by use of a quotient rule.

4. Differentiate $y = 4x^2$ by use of first principles.
5. Differentiate with respect to x:

   $$y = -\sqrt{3}\cos x + 2^{2x} - 4\ell nx - x^{\frac{2}{3}}$$

6. Given:

   $$y = x^3 - 6x^2 + 9x + 12$$

   Solve for the values of x if $\frac{dy}{dx} = 0$

7. Given:

   $$y = \frac{\sin 3x}{4x^3}$$

   Differentiate by using the quotient rule.

8. Determine:

   $$\lim_{x \to 2}\left(\frac{x^2 - x}{x^3 - x}\right)$$

9. Given:

   $$y = -2x^2 + 3x^2 + 12x - 5$$

   Determine, using differentiation, the co-ordinates of the maximum and minimum turning points.

10. Given:

    $$y = 10^x \tan x$$

    Differentiate by use of the product rule.

11. Differentiate the following with respect to x:

    $$y = \frac{1 - \cos x}{\sin x} - \frac{3}{\cos ecx} + \frac{3}{x} + \sqrt{x}$$

Answers:

1. $= 2x - 3$

   $= \frac{\cos x - \sin x \sec x \tan x}{\sec x}$

2. $f(x) = 4x^2$

3. $= 8x$

4. $\frac{dy}{dx} = \sqrt{3}\sin x + 2.2^x \ell n2 - \frac{4}{x} - \frac{2}{3}x^{-\frac{1}{3}}$

5. $x_1 = 3 \qquad x_2 = 1$

6. $\frac{dy}{dx} = \frac{(4x^3)(3\cos 3x) - (\sin 3x)(12x^2)}{(4x^3)^2}$

7. $= \frac{2}{7}$

8. $= -12$

9. $(2; 15)(-1; -12)$

10. $= 10^x \ell n10.\tan x + \sec^2 x.10^x$

11. $\frac{dy}{dx} = -\cos ecx \cot x + \cos ec^2 x - 3\cos x - \frac{3}{x^2} + \frac{1}{2}x^{-\frac{1}{2}}$

Self-Check

I am able to:	Yes	No
• Describe the binomial theorem in general terms		
• Expand a binomial to four terms		
• Calculate simple limits with the theorems on limits		
• Define differentiation and derive the expression for the rate of change		
• Differentiate to standard forms		
• Calculate by applying the chain rule to a function of a function		
• Determine the first derivatives of polynomials		
• Apply the product and quotient rules for differentiation		
• Determine the second derivatives of trigonometric functions		
If you have answered 'no' to any of the outcomes listed above, then speak to your facilitator for guidance and further development.		

Module 7

Integration

Learning Outcomes
By the end of the module you should be able to:

- Describe the concept of a definite integral and indefinite integral
- Apply standard forms of integrals as the converse of differentiation
- Integrate polynomials with terms
- Calculate definite integrals of functions with terms
- Calculate the area under a curve using integration

7.1 Introduction

Integration is the inverse process to differentiation. If the derivative of a function is given, we determine the original function.

For example, when we find the derivative of $f(x) = 6x$ we get:

$$f'(x) = 6$$

When we integrate $f'(x) = 6$ we get $6x$ … the original function.

Another example is if we integrate $f'(x) = 4x^3$ we ask, what did the original function look like to get the derivative $f'(x) = 4x^3$?

The answer is $f(x) = x^4$. This is the integral.

The process of integration has the symbol $\int$. This is known as the integral sign.

In the expression $\int f(x)\,dx$ the function $f(x)$ is called the integrand

The sign $\int$ *together with* dx is regarded as an operator

The operator indicates that the function $f(x)$ must be integrated with respect to x.

7.2 Indefinite integrals

Different functions have the same derivatives. Look at the example:

$$f(x) = x^2 \quad \rightarrow \quad f'(x) = 2x$$

$$f(x) = x^2 + 8 \quad \rightarrow \quad f'(x) = 2x$$

$$f(x) = x^2 - \pi \quad \rightarrow \quad f'(x) = 2x$$

$$\text{The indefinite integral } \int f(x)\ dx \text{ is not a unique function}$$

An arbitrary constant "c" is added so that an indefinite integral is always written:

$$\int f(x)\ dx + c$$

7.3 Standard forms of integrals

Say f and g are functions of x and c is the constant. Then:

1. $\int c\ f(x)\ dx \ = \ k \int f(x)\, dx$

2. $\int\ [f(x) \pm g(x)]\, dx \ = \ \int f(x)\, dx \pm \int g(x)\ dx$

3	$\int a\ dx = ax + c$ "a" any constant
4	$\int x^n\ dx = \frac{x^{n+1}}{n+1} + c \quad n \neq -1$
5	$\int \frac{1}{x}\ dx = \ln x + c$
6	$\int e^x\ dx = e^x + c$
7	$\int a^x\ dx = \frac{1}{\ln a} a^x + c$
8	$\int \cos x\ dx = \sin x + c$
9	$\int \sin x\ dx = -\cos x + c$
10	$\int \sec^2 x\ dx = \tan x + c$

11	$\int \operatorname{cosec}^2 x \;\; dx \;=\; -\; \cot x + c$
12	$\int \sec x \tan x \;\; dx \;=\; \sec x + c$
13	$\int \operatorname{cosec} x \cot x \;\; dx \;=\; -\operatorname{cosec} x + c$

Table 7.1 Standard forms of integrals

Worked Example 7.1

$$\text{Determine } \int \frac{5}{x} \; dx$$

Solution:

$$\int \frac{5}{x} \; dx \;=\; 5\int \frac{1}{x} \; dx \qquad \text{Rule 1.}$$

$$=\; 5 \; \ln x + c \qquad \text{Rule 5.}$$

Worked Example 7.2

$$\text{Determine } \int \frac{dx}{\sec x}$$

Solution:

$$\int \frac{dx}{\sec x} = \int \frac{1}{\sec x} \, dx = \int \cos x \;\; dx$$

$$= \;\; \sin x + k \quad \text{Rule 8.}$$

Worked Example 7.3

$$\text{Determine} \int \frac{1}{x^2}\ dx$$

Solution:

$$\int \frac{1}{x^2}\ dx \ = \ \int x^{-2}\ dx$$

$$= \frac{x^{-1}}{-1}$$

$$= -\ \frac{1}{x} + \ c$$

7.4 Polynomials with terms

7.4.1 Remember the chain rule

Because integration is the inverse of differentiation, let's look at the chain rule again.

$$If \quad y = \sin x^2$$

$$Then \quad y' = \cos x^2 \times 2x$$

$$= 2\text{x} \cos x^2$$

$$\int 2\text{x} \cos x^2\ dx \ = \ \sin x^2 + c$$

This gives rise to another special rule:

14. $\int f'(x)\ \cos f(x)\ dx \ = \ \sin f(x) + c$

Worked Example 7.4

$$\text{determine} \int 3x^2 \cos x^3\ dx$$

Solution:

$$\int 3x^2 \cos x^3\ dx \ = \ 3x^2\ \cos x^3\ dx \ = \ \sin f(x) + c$$

$$= \ \sin x^3 + c$$

7.4.2 Other integrals

15. $\int f'(x)\ \sec^2 f(x)\ dx\ =\ \tan\ f(x) + c$

Worked Example 7.5

$$\text{determine} \int 2x \sec^2 x^2\ dx$$

Solution:

$$\int 2x \sec^2 x^2\ dx\ =\ \int f(x)\ \sec^2 f(x)\ dx$$

$$= \tan f(x) + c$$

$$= \tan x^2 + c$$

16. $\int f'(x)\ [f(x)]^n\ dx\ =\ \dfrac{[f(x)]^{n+1}}{n+1} + c$

Worked Example 7.6

$$\text{determine} \int 6x\ (3^2 + 7)^5\ dx$$

Solution:

$$\int 6x\ (3^2 + 7)^5\ dx\ =\ \int f'(x)\ [\ f(x)]^5\ dx$$

$$= \frac{[f(x)]^{5+1}}{5+1} + c$$

$$= \frac{(3x^2+7)^6}{6} + c$$

Worked Example 7.7

Integrate the following:

$$\int \left(-4x^{-2} - \frac{4}{x} - \frac{e^x}{4} - 10^{4x} + \frac{1}{\sqrt{x}} + 3\cos ecx \cot x\right) dx$$

Solution:

$$\frac{4}{x} - 4\ell nx - \frac{e^x}{4} - \frac{10^{4x}}{4\ell n10} + \sqrt[2]{x} - 3\cos ecx + c$$

Worked Example 7.8

Simplify:

$$\int \left(\frac{x^3 - 27}{x - 3}\right) dx$$

Solution:

$$\frac{x^3}{3} + \frac{3x^2}{2} + 9x + c$$

7.5 Definite integrals

Definition:
A Definite Integral has start and end values: in other words, there is an interval (a to b). The values are put at the bottom and top of the symbol.

An example is when integration is used to find the area under a curve. When the area has boundaries, say between *a* and *b* as in **Figure 7.1**. then the start and end values are placed on the symbol as shown below:

$$\int_a^b f(x)\, dx$$

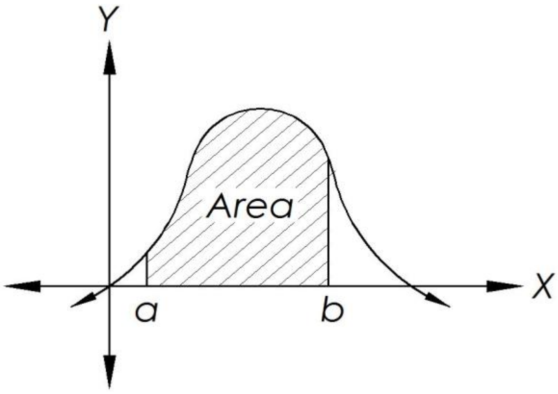


Figure 7.1 Definite integral has boundaries

Worked Example 7.9

Integrate the following:

$$\int_1^2 2x \ dx$$

Solution:

Because the symbol has boundaries $\int_1^2$ we know that we are dealing with a definite integral.

In order to work this out:

- Create a rough sketch of the curve
- Find the indefinite integral first

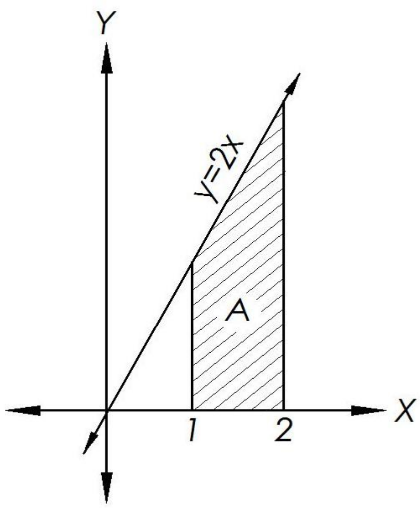


Figure 7.2 Straight line curve with boundaries between 1 and 2

The indefinite integral:

$$\int_{1}^{2} 2x \ dx \quad = \quad x^2 + c$$

When $x = 1$ Then $\int_{1}^{2} 2x \ dx \quad = \quad 1^2 + c$

When $x = 2$ Then $\int_{1}^{2} 2x \ dx \quad = \quad 2^2 + c$

Subtract:

$$(2^2 + c) \ - \ (1^2 + c)$$

$$= \ 2^2 - \ 1^2 + c - c$$

$$= \ 4 - \ 1 \quad = \quad 3$$

The "c" gets cancelled out, so with definite integrals the constant, "c" can be ignored.

In fact, we can simply solve it this way:

$$\int_{1}^{2} 2x \ dx \ = \ 2^2 - 1^2 \ = \ 3$$

Check the answer by finding the area that's hatched in **Figure 7.2**.

Worked Example 7.10

$$Find\ the\ area\ under\ the\ curve \quad y = x^3 + 2:\ \ 1 \leq x \leq 4$$

Solution:

$$A = \int_a^b y \ \ dx$$

$$= \int_1^4 (x^3 + 2)\ \ dx$$

$$= \left[\frac{x^4}{4} + 2x\right]_1^4$$

$$= \left[\frac{x^4}{4} + 2.4\right] - \left[\frac{1^4}{4} + 2.1\right]$$

$$= \ 69.75 \text{ square units}$$

Worked Example 7.11

Evaluate:

$$\int_1^2 (5x^4 + 4x^3) dx$$

Solution:

$$\int_1^2 (5x^4 + 4x^3) dx$$

$$= [x^5 + x^4]^2$$

$$= [(2^5 + 2^4) - (1^5 + 1^4)]$$

$$= 48 - 2$$

$$= 46$$

Worked Example 7.12

1. Sketch the graph of $y = 5\sin x$, $x = 0$ and $x = \frac{n}{2}$ and indicate the enclosed area. Also, indicate the representative strip used to calculate the enclosed area.
2. Calculate the value of the indicated area by using integration.

Solution:

1.

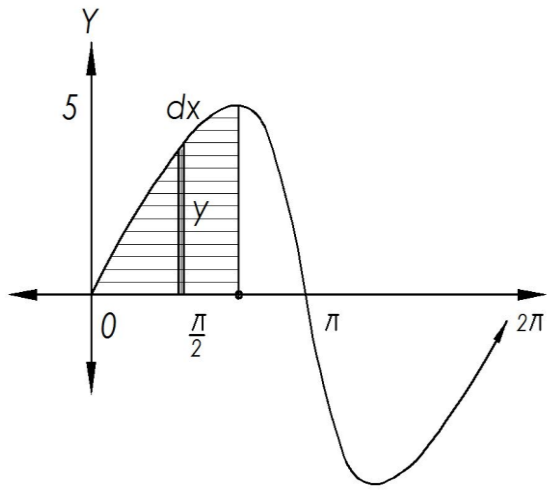


Figure 7.3

2.

$AoX = \int_1^2 (5x^4 + 4x^3)dx$

$= [-5\cos x]_0^{\frac{n}{2}}$

$= \left[\left(-5\cos\frac{n}{2} - (-5\cos 0)\right)\right]$

$= 0 + 5$

$= 5u^2$

Activity 7.1

1. $\int 3x^5 dx$

2. $\int \sqrt[3]{2x}\, dx$

3. $\int \frac{(1+y)^2}{\sqrt{y}} dy$

4. $\int 3x\, dx$

Answers:

1. $\frac{1}{2}x^6 + k$

2. $\frac{3\sqrt{2}\times\frac{4}{3}}{4} + k$

3. $2\sqrt{y} + \frac{4}{3}y^{\frac{3}{2}} + \frac{2}{5}y^{\frac{5}{2}} + k$

4. $\frac{3}{2}x^2 + k$

Activity 7.2

1. $\int 5x^4 \cos x^3\, dx$

2. $\int 3x^2 \sec^2 x^3\, dx$

3. $\int 2x(x^2 + 5)^{10} dx$

4. $\int (2x + x)(x^2 + 2x - 3)^7 dx$

Answers:

1. $\sin x^5 + k$

2. $tax\, x^3 + k$

3. $\frac{1}{11}(x^2 + 5)^{11} + k$

4. $\frac{1}{8}(x^2 + 2x - 3)^8 + k$

Activity 7.3

Integrate the following:

1. $\int \left(8e^{4t} - 3\cos 3t + \frac{1}{\sqrt{t}} - 11^{2t} + \sqrt[3]{t} + 2\sec^2 t\right) dt$

2. $\int \left(\sin 5x + \sec^2 x - 3^x + 9e^{-3x} + 4x^{\frac{1}{4}}\right) dx$

3. $\int \left(-4x^3 - 9^{-3x} + 14\sin 7x - \frac{3}{x} + \sec^2 4x - 4\right) dx$

4. $\int \left(5x^5 - 2^{10x} + 4e^{3x} - \frac{4}{\cos ec\, 4x} - \frac{1}{x} + a\right) dx$

Answers:

1. $= -2e^{4t} - \sin 3t + \sqrt{t} - \frac{11^{2t}}{2\ell n 11} + 2\sqrt[3]{t^2 + 2\tan t + c}$

2. $-\frac{\cos 5x}{5} + \tan x - \frac{3^x}{\ell n 3} - 3e^{-3x} + \frac{16}{5}x^{\frac{5}{4}} + c$

3. $\frac{-4x^4}{4} - \frac{9^{-3x}}{(-3)\ell n 9} - \frac{14\, sin^{7x}}{7} - 3\ell n x + \tan x - 4x + c$

4. $\frac{3x^6}{6} - \frac{2^{10x}}{2\ell n 10} + \frac{4e^{3x}}{3} + \frac{4\cos 4x}{4} - \ell n x + ax + c$

Activity 7.4

Simplify the following:

1. $\int \sqrt{\cos^2 x + \sin^2 x}\, dx$

2. $\int (4\cos 4x + 3) dx$

3. $\int \sqrt{x^2 - 6x + 9}\, dx$

4. $\int 2ax^4 dx$

Answers:

1. $x + c$

2. $\sin 4x + 3x + c$

3. $= -0{,}384 \quad = 0{,}384 \quad x = 2 \quad = 0$

4. $= \frac{2ax^5}{5} + c$

Activity 7.5

Evaluate the following:

1. $\int_2^3 \frac{dt}{t}$

2. $\int_0^1 2^{-3x} dx$

3. $\int_0^{\frac{n}{4}} (4 \cos 2x) dx$

4. $\int_1^2 2^{-5x} dx$

Answers:

1. $1.0986 - 0.6931$

2. $= 0{,}421$

3. $= 2\ units$

4. $= 0{,}00901\ units$

Activity 7.6

1.
    1.1 Sketch and indicate the area enclosed by the graph of $y = 4x$ with $x = 2$ and $x = 3$. Also indicate the representative strip to be used to calculate the area enclosed.
    1.2 Calculate, using integration, the area indicated in Question 1.1.
2.
    2.1 Sketch and indicate the area enclose by the graph of $y = 4 \cos x$ with $x = 0 \circ$ and $x = 90 \circ$. Also indicate the representative strip to be

used to calculate the area enclosed.

2.2 Calculate, using Integration, the value of the area indicated in Question 2.1.

3.

3.1 Sketch and indicate the area enclosed by the graph of $y = 3\sin x$ the x-axis; $x = 0$ and $\frac{\pi}{2}$. Also, show the representative strip to be used to calculate the area enclosed.

3.2 Calculate, using Integration, the magnitude of the area indicated in QUESTION 3.1.

4.

4.1 Sketch and indicate the area enclosed by the graph of $y = 3x - x^2$ the X-axis. Also, indicate the representative strip to be used to calculate the value of the enclosed area.

4.2 Calculate, using Integration, the value of the enclosed area in QUESTION 4.1.

Answers:

1.1

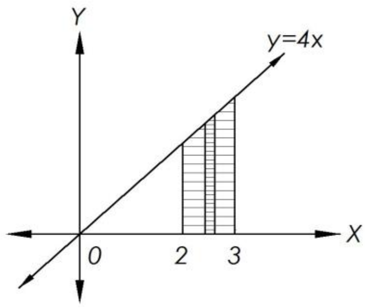


Figure 7.4

1.2 $Aox = \int y dx$

$= \int 4x dx$

$Aox = \int_2^3 4x dx$

$= [2x^2]_2^{\ 3} = [2(3)^2 - 2(2)^2] = 10u$

2.1

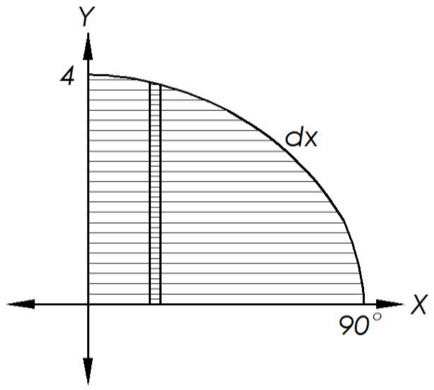


Figure 7.5

2.2 $AOX = ydx$

$= 4\cos xdx$

$AOX = \int_0^{90°} 4\ \cos xdx$

$= [4\sin x]_0^{90°}$

$= 4u^2$

3.1

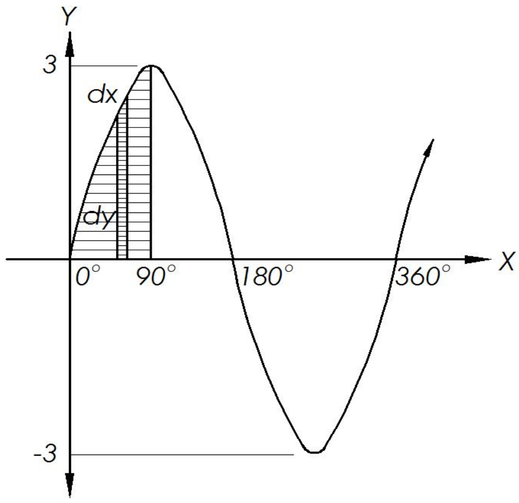


Figure 7.6

3.2 $AOX = \int_0^{\frac{n}{2}} 3\ \sin xdx$

$= [-3\cos x]_0^{\frac{n}{2}}$

$= [0] - [-3]$

$= 3u^2$

4.1

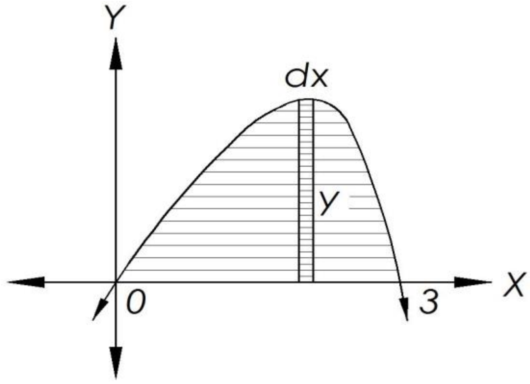


Figure 7.7

4.2 $AOX = ydx$

$= 4\cos xdx$

$AOX = \int_0^3 (3x - x^2)dx$

$= \left[\frac{3x^2}{2} - \frac{x^3}{3}\right]_0^3$

$= \left[\frac{3(3)^2}{2} - \frac{3^3}{3}\right] - [0]$

$= 4,5\ u^2$

Self-Check

I am able to:	**Yes**	**No**
• Describe the concept of a definite integral and indefinite integral		
• Apply standard forms of integrals as the converse of differentiation		
• Integrate polynomials with terms		
• Calculate definite integrals of functions with terms		
• Calculate the area under a curve using integration		
If you have answered 'no' to any of the outcomes listed above, then speak to your facilitator for guidance and further development.		

Past Examination Papers

higher education & training
Department:
Higher Education and Training
REPUBLIC OF SOUTH AFRICA

NOVEMBER 2014

NATIONAL CERTIFICATE

MATHEMATICS N4

(16030164)

14 November 2014 (Y-Paper)
13:00 – 16:00

Calculators may be used.

This question paper consists of 5 pages and 1 formula sheet.

DEPARTMENT OF HIGHER EDUCATION AND TRAINING
REPUBLIC OF SOUTH AFRICA
NATIONAL CERTIFICATE
MATHEMATICS N4
TIME: 3 HOURS
MARKS: 100

INSTRUCTIONS AND INFORMATION

1. Answer ALL the questions.
2. Show ALL the calculations and intermediary steps. Simplify where possible.
3. All the graph work must be done in the ANSWER BOOK. Graph paper is NOT supplied. Values of intercepts with the system of axes and the turning point(s) MUST be shown on the graph.
4. ALL final answers must be accurately approximated to THREE decimal places.
5. Questions may be answered in any order but subsections of questions must NOT be separated.
6. A formula sheet is attached to this question paper. You are NOT compelled to use the formulae and the list is NOT necessarily complete.
7. Number the answers according to the numbering system used in this question paper.
8. Write neatly and legibly.

QUESTION 1:

1.1 The area of a rectangle is 54 cm^2. The length of the rectangle is 3 cm more than its breadth. Determine the length and the breadth of the rectangle. (5)

1.2 Solve for x if: (4)

1.3 Make q the subject of the formula: (3)

$$Z\ell n\left(\frac{P}{q}\right) = w$$

1.4 Given: (8)

$$2p + 3q = r - 3$$
$$-2q - p = 7 + 3r$$
$$-3r + 2q = -2p + 9$$

Solve the value of q

[20]

QUESTION 2:

2.1 Simplify and leave answer in $a + jb$ form. (4)

$$\frac{2\underline{|50°}.3\underline{|40°}}{4\underline{|30°}.3\underline{|-10°}}$$

2.2 Solve for x and y: (4)

$$(1 - 2j)^2 = x + yj$$

2.3 Simplify: (2)

$$\frac{j^{100}}{j^{20}}$$

2.4 2.4.1 Sketch the graph of $x^2 + 2y^2 = 2$ (3)

2.4.2 Is the graph $x^2 + 2y^2 = 2$ symmetric of asymmetric about the X-axis? (1)

2.4.3 What is the domain of the graph of $x^2 + 2y^2 = 2$ (2)

2.5 2.5.1 Sketch the graph of $y = 5e^x$ (2)

2.5.2 Is the graph of $y = 5e^{x}$ continuous or discontinuous? (1)

2.5.3 Is the graph of $y = 5e^{x}$ a function? (1)

[20]

QUESTION 3:

3.1 3.1.1 Derive a formula for $\cos\frac{x}{2}$ if $\cos 2x = 2cos^{2}x - 1$ (3)

3.1.2 Use the derived formula in QUESTION 3.1.1 to calculate the value of $\cos 15$ without the use of the calculator. (3)

3.2 Solve for x if: (4)

$$sec^{2}x + tan^{2}x = \frac{5}{3}; 0^{0} \leq x \leq \pi$$

3.3 Prove that: (4)

$$(cosec\ x + \cot x)^{2} = \frac{1 + \cos x}{1 \cos x}$$

3.4 Simplify as far as possible: (4)

$$(\tan x + \sec x)(1 - \sin x)$$

3.5 Derive a formula for $\cos 2B$ in terms of $\sin B$. (2)

[20]

QUESTION 4:

4.1 Determine the limit of the following: (4)

$$\lim_{x \to 2}\left(\frac{2x^{2} - 8}{x - 2}\right)$$

4.2 Differentiate by the use of a Product Rule if: (4)

$$y = 4^{x}\ cosec\ x$$

4.3 Given: $y = x^{3} + 5x^{2} + 6x\ \ 3$ (8)

Determine, using differentiation, the coordinates of the maximum and minimum turning points.

4.4 Differentiate the following: (4)

$$y = \frac{3}{\sin x} - 2^{2x} + 6e^{3x} + x^3$$

[20]

QUESTION 5:

5.1 Integrate the following: (7)

$$\int \left(sec^2 7x - 3 + \frac{3}{x^3 + 3\cos ecx \cot x - 3^{-3x} - 3e^{-3x}}\right) dx$$

5.2 5.2.1 Sketch and indicate the area enclosed by the graph of $y = 2.3^{2x}$, the X-axis, the Y-axis and $x = 3$. Also, indicate the representative strip to be used to calculate the enclosed area. (3)

5.2.2 Calculate, using integration, the value of the enclosed area indicated in QUESTION 5.2.1. (4)

5.3 Determine: (3)

$$\int_{\circ}^{n/4} 2\sin 2\theta \, d\theta$$

5.4 Simplify: (3)

$$\int \frac{3}{\cot x \cos x} dx$$

[20]

TOTAL: 100

MATHEMATIC N4

FORMULA SHEET

NEW SYLLABUS

$a^x = b \Leftrightarrow \log a^x = \log b$

$\ell n\, x = \log_e x$

$(r\,|\underline{\theta})^n = r^n\,|\underline{n\theta}$ $\quad a + bj = c + dj \Leftrightarrow a = c$ and $b = d$

$\sin(a \pm b) = \sin a \cos b \pm \sin b \cos a$
$\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$

$\sin^2 x + \cos^2 x = 1$
$1 + \cot^2 x = \operatorname{cosec}^2 x$
$1 + \tan^2 x = \sec^2 x$

$$\tan(a \pm b) = \frac{\tan a \pm \tan b}{1 \mp \tan a \tan b}$$

y	$\frac{dy}{dx}$
ax^n	nax^{n-1}
ka^x	$ka^x \ell na$
$k\ell nx$	$\frac{k}{x}$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\sec x$	$\sec x \tan x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$

$y = u(x) \cdot v(x)$

$$\Rightarrow \frac{dy}{dx} = u(x)v'(x) + u'(x)\,v(x)$$

$$y = \frac{u(x)}{v(x)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{v(x)u'(x) - u(x)v'(x)}{[v(x)]^2}$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\int ax^n\,dx = \frac{ax^{n+1}}{n+1} + C$$

$$\int \sin x\,dx = -\cos x + c$$

$$\int \frac{a}{x}\,dx = a\ell n\,x + c$$

$$\int \cos x\,dx = \sin x + c$$

$$\int ka^x dx = \frac{ka^x}{\ell na} + c$$

$$\int \tan x\,dx = \ell n \sec x + c$$

$$\int \sec x\,dx = \ell n(\sec x + \tan x) + c$$

$$A_{ox} = \int_a^b y\,dx$$

Marking Guidelines

higher education & training

Department:
Higher Education and Training
REPUBLIC OF SOUTH AFRICA

NOVEMBER 2014

NATIONAL CERTIFICATE

MATHEMATICS N4

(16030164)

14 November 2014 (Y-Paper)
13:00 – 16:00

QUESTION 1

1.1 $\left.\begin{array}{l}\ell = b + 3 \ldots\ldots\ldots..(1) \\ \ell . b = 54 \ldots\ldots\ldots..(2)\end{array}\right\}$ √

In (2) $(b+3).b = 54$ √

$b^2 + 3b - 54 = 0$ √

$(b+9)(b-6) = 0$

$b_1 = -9$ and $b_2 = 6$ m √

In (1) $\ell = 6 + 3$

$= 9$ m √ (5)

1.2 $\ell n3 + (x+2)\ell n6 = (x-1)\ell n14$ √

$1{,}0986 + 1{,}791x + 3{,}584 = 2{,}639x - 2{,}369$ √

$-0{,}848x = -7{,}322$ √

$x = 8{,}634$ √ (4)

1.3 $\ell n\left(\frac{p}{q}\right) = \frac{w}{z}$ √

$\frac{p}{q} = e^{\frac{w}{z}}$ √

$q = \frac{p}{e^{\frac{w}{z}}}$ √

(3)

1.4

$$|D| = \begin{vmatrix} 2 & 3 & -1 \\ -1 & -2 & -3 \\ 2 & 2 & -3 \end{vmatrix} \quad \surd$$

$$|D| = 2\begin{vmatrix} -2 & -3 \\ 2 & -3 \end{vmatrix} \quad -3\begin{vmatrix} -1 & -3 \\ 2 & -3 \end{vmatrix} \quad -1\begin{vmatrix} -1 & -2 \\ 2 & 2 \end{vmatrix} \quad \surd$$

$$|D| = 2(12) - 3(9) - 1(2) \quad \surd$$

$$|D| = -5 \quad \surd$$

$$|Dq| = \begin{vmatrix} 2 & -3 & -1 \\ -1 & 7 & -3 \\ 2 & 9 & -3 \end{vmatrix} \quad \surd$$

$$|Dq| = 2\begin{vmatrix} 7 & -3 \\ 9 & -3 \end{vmatrix} \quad +3\begin{vmatrix} -1 & -3 \\ 2 & -3 \end{vmatrix} \quad -1\begin{vmatrix} -1 & 7 \\ 2 & 9 \end{vmatrix} \quad \surd$$

$$|Dq| = 2(6) + 3(9) - 1(-23)$$

$$|Dq| = 62 \quad \surd$$

$$q = \frac{|D_q|}{D} = \frac{62}{-5} = 12{,}4 \quad \surd$$

(8)

[20]

QUESTION 2

2.1

$$= \frac{6\angle 90^0}{12\angle 20^0} \quad \surd$$

$$= 0{,}5\angle 70^0 \quad \surd$$

$$= 0{,}5\left[\cos 70^0 + j\sin 70^0\right] \quad \surd$$

$$= 0{,}171 + j0{,}469 \quad \surd$$

(4)

2.2

$$1 - 2j - 2j + 4j^2 = x + yj \quad \surd$$

$$1 - 4j - 4 = x + yj \quad \surd$$

$$x + yj = -3 - 4j$$

$$x = -3 \quad \surd$$

$$y = -4 \quad \surd$$

(4)

2.3

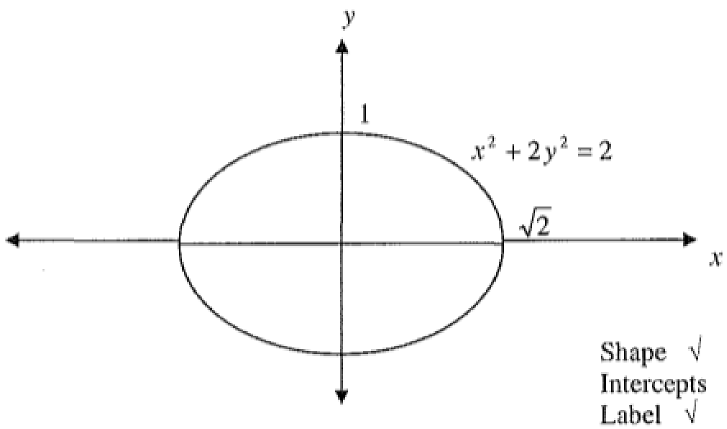
$$\frac{j^{2(50)}}{j^{2(10)}}$$

$$= \frac{(-1)^{50}}{(-1)^{20}} \quad \surd$$

$$= 1 \quad \surd$$

(2)

2.4 2.4.1



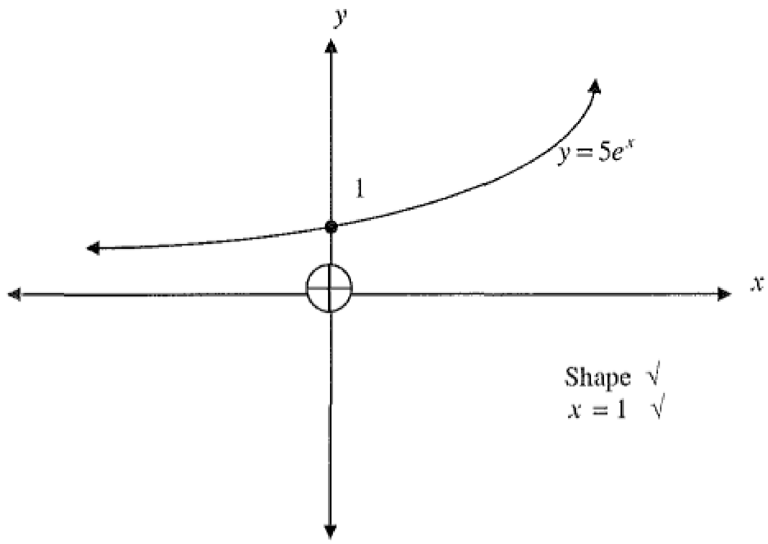
Shape √
Intercepts √
Label √

(3)

2.4.2 Symmetric √ (1)

2.4.3 $\{x : -\sqrt{2} \le x \le \sqrt{2}, xE\}$ √ √ (2)

2.5 2.5.1



Shape √
$x = 1$ √

(2)

2.5.2 Continuous √ (1)

2.5.3 Yes (1)

[20]

QUESTION 3

3.1 3.1.1 $2\cos^2 x = \cos 2x + 1.$

$\cos^2 = \dfrac{\cos 2x + 1}{2}$ √

$\cos x = \sqrt{\dfrac{\cos 2x + 1}{2}}$

$\cos\dfrac{x}{2} = \sqrt{\dfrac{\cos\left(2\,\frac{x}{2}\right) + 1}{2}}$ √

$\cos\dfrac{x}{2} = \sqrt{\dfrac{\cos x + 1}{2}}$ √ (3)

3.1.2 $\cos 15^0 = \sqrt{\dfrac{\cos 30^0 + 1}{2}}$ √

$= \sqrt{\dfrac{\frac{\sqrt{3}}{2} + 1}{2}}$ √

$\cos 15^0 = \dfrac{3 + \sqrt{2}}{2}$ √ (3)

3.2 $1 + \tan^2 x + \tan^2 x = \frac{5}{3}$ √

$2\tan^2 x = 0{,}667$

$\tan^2 x = 0{,}334$ √

$\tan x = 0{,}578$ √

$x_1 = 30{,}0279^0$ √ (4)

3.3 $\left(\frac{1}{\sin x}+\frac{\cos x}{\sin x}\right)^2=\frac{1+\cos x}{1-\cos x}$

$\left(\frac{1+\cos x}{\sin x}\right)^2=\frac{1+\cos x}{1-\cos x}$ √

$\frac{(1+\cos x)^2}{\sin^2 x}=\frac{1+\cos x}{1-\cos x}$ √

$\frac{(1+\cos \mathrm{x})^2}{1 \quad \cos^2 \mathrm{x}}=\frac{1+\cos \mathrm{x}}{1 \quad \cos \mathrm{x}}$

$\frac{(1+\cos x)(1+\cos x)}{(1-\cos x)(1+\cos x)}=\frac{1+\cos x}{1-\cos x}$ √

$\frac{1+\cos x}{1-\cos x}=\frac{1+\cos x}{1-\cos x}$ √ (4)

3.4 $=\left(\frac{\sin x}{\cos x}+\frac{1}{\cos x}\right)(1-\sin x)$

$=\left(\frac{1+\sin x}{\cos x}\right)(1-\sin x)$ √

$=\frac{1-\sin^2 x}{\cos x}$ √

$=\frac{\cos^2 x}{\cos x}$ √

$=\cos x$ √ (4)

3.5 $\cos 2B=\cos B\cos B-\sin B\sin B$

$=\cos^2 B-\sin^2 B$ √

$\text{Cos2 B} = 1 - \sin^2 - \sin^2 \text{B}$

$\therefore \cos 2\text{B} = 1 - 2\sin^2 \text{B}$ √ (2)

[20]

3.3

$$\left(\frac{1}{\sin x}+\frac{\cos x}{\sin x}\right)^2=\frac{1+\cos x}{1-\cos x}$$

$$\left(\frac{1+\cos x}{\sin x}\right)^2=\frac{1+\cos x}{1-\cos x} \quad \surd$$

$$\frac{(1+\cos x)^2}{\sin^2 x}=\frac{1+\cos x}{1-\cos x} \quad \surd$$

$$\frac{(1+\cos \mathrm{x})^2}{1 \quad \cos^2 \mathrm{x}}=\frac{1+\cos \mathrm{x}}{1 \quad \cos \mathrm{x}}$$

$$\frac{(1+\cos x)(1+\cos x)}{(1-\cos x)(1+\cos x)}=\frac{1+\cos x}{1-\cos x} \quad \surd$$

$$\frac{1+\cos x}{1-\cos x}=\frac{1+\cos x}{1-\cos x} \quad \surd$$

(4)

3.4

$$=\left(\frac{\sin x}{\cos x}+\frac{1}{\cos x}\right)(1-\sin x)$$

$$=\left(\frac{1+\sin x}{\cos x}\right)(1-\sin x) \quad \surd$$

$$=\frac{1-\sin^2 x}{\cos x} \quad \surd$$

$$=\frac{\cos^2 x}{\cos x} \quad \surd$$

$$=\cos x \quad \surd$$

(4)

3.5

$$\cos 2B=\cos B\cos B-\sin B\sin B$$

$$=\cos^2 B-\sin^2 B \quad \surd$$

$$\mathrm{Cos2\ B} = 1 - \sin^2 - \sin^2 \mathrm{B}$$

$$\therefore \mathrm{cos2B} = 1 - 2\sin^2 \mathrm{B} \quad \surd$$

(2)

[20]

QUESTION 4

4.1 $\lim_{x \to 2} 2\left[\frac{(x-2)(x+2)}{x-2}\right]$ √

$= 2(x+2)$ √

$= 2x+4$ √

$= 8$ √ (4)

4.2 $u = 4^x ; \frac{du}{dx} = 4^x \ell n4$ √

$V = \cos ecx \frac{dv}{dx} = -\cos ecx \cot c$ √

$\frac{dy}{dx} = 4^x . -\cos ecx \cot x + \cos ecx . 4^x \ell n4$ √

$= 4^x \cos ecx(-\cot x + \ell n4)$ √ (4)

4.3 $\frac{dy}{dx} = 3x^2 + 10x + 6$ √

$3x^2 + 10x + 6 = 0$ √

$x = \frac{-10 \pm \sqrt{(10)^2 - 4(3)(6)}}{6}$

$x_1 = -0{,}785$ √ and/en $x_2 = -2{,}545$ √

$y_1 = (-0{,}785)^3 + 5(-0{,}785)^2 + 6(-0{,}785) - 3$

$y_1 = -5{,}113$ √

$y_2 = (-2{,}545)^3 + 5(-2{,}545)^2 + 6(-2{,}545) - 3$

$y_2 = -2{,}369$ √

$(-0{,}785;-5{,}113)$ √ $(-2{,}545;-2{,}369)$ √ (8)

4.4 $\frac{dy}{dx} = -3\cos ecx \cot x - 2^{2x}.2\ell n2 + 18e^{3x} + 3x^{2}$

√ √ √ √ (4)

[20]

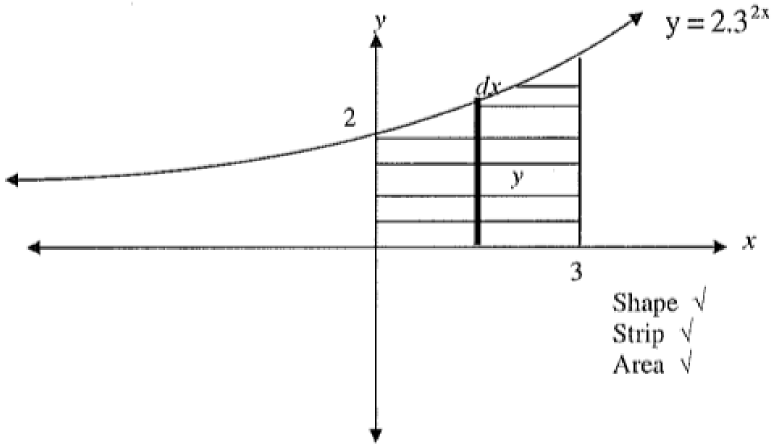
QUESTION 5

5.1 $\frac{\tan 7x}{7} - 3x - \frac{2}{3}x^{-2} - 3\cot x - \frac{3^{-3x}}{-3\ell n3} - \frac{3e^{-3x}}{-3} + c$

$= \frac{\tan 7x}{7} \quad 3x \quad \frac{2}{3}x^{\ 2} \quad 3\cot x + \frac{3^{\ 3x}}{3\ell n3} + e^{\ 3x} + c$

√ √ √ √ √ √ √ (7)

5.2 5.2.1



Shape √
Strip √
Area √

(3)

5.2.2 $AoX = \int_{0}^{3} 2.3^{2x}\,dx$ √

$= 2\left[\frac{3^{2x}}{2\ell n3}\right]_{0}^{3}$ √

$= 2\left[\frac{3^{2(3)}}{2\ell n3} - \frac{3^{2(0)}}{2\ell n3}\right]$ √

$= 662{,}654u^{2}$ √ (4)

5.3 $\left[\frac{-2\cos 2\theta}{2}\right]_0^{\pi/4}$ √

$-[\cos 2(\pi/4) - \cos 2(0)]$ √

- (- 1)

= 1 unit √ (3)

5.4 $\int 3\sec x \tan x dx$ √

$3\sec x + c$ √√ (3)

TOTAL: 100

Past Examination Papers

higher education & training

Department: Higher Education and Training
REPUBLIC OF SOUTH AFRICA

AUGUST 2014

NATIONAL CERTIFICATE

MATHEMATICS N4

(16030164)

24 July 2014 (Y-Paper)
13:00 – 16:00

Calculators may be used.

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DEPARTMENT OF HIGHER EDUCATION AND TRAINING
REPUBLIC OF SOUTH AFRICA
NATIONAL CERTIFICATE
MATHEMATICS N4
TIME: 3 HOURS
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1. Answer ALL the questions.
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6. ALL final answers must be accurately approximated to THREE decimal places.
7. Questions may be answered in any order but subsections of questions must NOT be separated.
8. A formula sheet is attached to this question paper. You are NOT compelled to use the formulae and the list is NOT necessarily complete.
9. Write neatly and legibly.

QUESTION 1:

1.1 Solve for x if: (3)

$5^{2x-1} = 7^{3x+4}$

1.2 Make D the subject of the formula if: (3)

$P = A \log\left(\frac{D-R}{R}\right)$

1.3 Given: (8)

$m - n + r + 3 = 0$
$2n + 4 - m = r$
$\frac{m}{2} - 1 = \frac{r}{3}$

1.3.1 Solve for m by the use of Cramer's Rule. (8)

1.3.2 Determine the value of the minor of -3. (2)

1.4 A steel wire of 28 m is bent to form a rectangle with an area of 46 m^2. (4)

Calculate the breadth and the length of the rectangle.

[20]

QUESTION 2:

2.1 Solve for a if: (5)

$2a^2 + 2a + 1 = 0$

2.2 Simplify and leave answer in $a + jb$ form (5)

$\frac{(4-2j)(5+j)}{3-2j}$

2.3 Sketch the graph of the inverse of (3)

$y = x - 6$

2.4 Sketch the graph of $y = \cos ecx$; $-180° \leq x \leq 180°$ (3)

2.5 2.5.1 Sketch the graph of $y = 13^x$. (3)

2.5.2 Is the graph of $y = 13^x$ in QUESTION 2.5.1 a function or a relation? (1)

[20]

QUESTION 3:

3.1 Calculate the value of $\tan 75°$ without the use of a calculator. (4)

3.2 Simplify: (4)

$$\frac{sin^2\theta}{1+\cos\theta}-1$$

3.3 Prove that: (4)

$$2\tan y=\frac{\sin 2y}{1+tan^2y}$$

3.4 Solve for A if: (4)

$$cos(A+10°)=sin(3A+20°);0°\leq A\leq 90°$$

3.5 If $\sin x=\frac{3}{5}$ and $\cos y=\frac{9}{41}$, with both x and y acute angles, calculate without the use of a calculator, the value of $\sin(x-y)$. (4)

[20]

QUESTION 4:

4.1 Given: $(2x^2-3)^5$ (4)

Expand to FOUR terms only by the use of the Binomial Theorem.

4.2 Differentiate the following: (4)

$$y=\frac{1}{3}\ell nx-3x^{-1}+\frac{5\sin 2x+7}{5\sin x}$$

4.3 Differentiate by the use of a Quotient Rule if: (4)

$$y=\frac{\sec x}{\log x}$$

4.4 Given: $y=-2x^3+3x^2+12x-7$ (8)

Determine, using differentiation, the maximum and the minimum turning points.

[20]

QUESTION 5:

5.1 Simplify: (3)

$$\int \frac{\tan x}{\sec x}\,dx$$

5.2 Determine: (3)

$$\int_{\circ}^{n/3} (4\cos 4x)\,dx$$

5.3 5.3.1 Sketch and indicate the area enclosed by the graph of $y = -x^2 + 2x$ and the X-axis. Also indicate the representative strip used to calculate the area enclosed. (3)

5.3.2 Calculate, using Integration, the value of the enclosed area. (4)

5.4 Integrate the following: (7)

$$\int \left(\frac{1}{3}\sec x\tan x - \frac{1}{3}e^{-2x} + \sqrt{x} + 4x^{-5} - 2.4^{3x} + p\right)dx$$

[20]

TOTAL: 100

MATHEMATIC N4

FORMULA SHEET

NEW SYLLABUS

$a^x = b \Leftrightarrow \log a^x = \log b$

$\ell n\, x = \log_e x$

$(r\,|\underline{\theta})^n = r^n\,|\underline{n\theta}$ $\quad a + bj = c + dj \Leftrightarrow a = c$ and $b = d$

$\sin(a \pm b) = \sin a \cos b \pm \sin b \cos a$
$\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$

$\sin^2 x + \cos^2 x = 1$
$1 + \cot^2 x = \mathrm{cosec}^2 x$
$1 + \tan^2 x = \sec^2 x$

$$\tan(a \pm b) = \frac{\tan a \pm \tan b}{1 \mp \tan a \tan b}$$

y	$\frac{dy}{dx}$
ax^n	nax^{n-1}
ka^x	$ka^x \ell na$
$k\ell nx$	$\frac{k}{x}$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\cot x$	$-\mathrm{cosec}^2 x$
$\sec x$	$\sec x \tan x$
$\mathrm{cosec}\, x$	$-\mathrm{cosec}\, x \cot x$

$y = u(x) \cdot v(x)$

$$\Rightarrow \frac{dy}{dx} = u(x)v'(x) + u'(x)\,v(x)$$

$$y = \frac{u(x)}{v(x)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{v(x)u'(x) - u(x)v'(x)}{[v(x)]^2}$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\int ax^n\,dx = \frac{ax^{n+1}}{n+1} + C$$

$$\int \sin x\,dx = -\cos x + c$$

$$\int \frac{a}{x}\,dx = a\ell n\, x + c$$

$$\int \cos x\,dx = \sin x + c$$

$$\int ka^x dx = \frac{ka^x}{\ell na} + c$$

$$\int \tan x\,dx = \ell n \sec x + c$$

$$\int \sec x\,dx = \ell n\,(\sec x + \tan x) + c$$

$$A_{ox} = \int_a^b y\,dx$$

Marking Guidelines

higher education & training

Department:
Higher Education and Training
REPUBLIC OF SOUTH AFRICA

AUGUST 2014

NATIONAL CERTIFICATE

MATHEMATICS N4

(16030164)

24 July 2014 (Y-Paper)
13:00 – 16:00

QUESTION 1

1.1 $(2x-1)\ell n5=(3x+4)\ell n7$ √

$3{,}219x-1{,}609=5{,}838x+7{,}784$ √

$-2{,}619x = 9{,}393$

$x = -3{,}586$ √ (3)

1.2 $\ell og\left(\frac{D-R}{R}\right)=\frac{P}{A}$ √

$\frac{D-R}{R}=10^{\frac{P}{A}}$ √

$D-R=R10^{\frac{P}{A}}$

$D=R10^{\frac{P}{A}}+R$ √ (3)

1.3 1.3.1

$|D|=\begin{vmatrix}1 & -1 & 1\\ -1 & 2 & -1\\ \frac{1}{2} & 0 & \frac{-1}{3}\end{vmatrix}$ √ $\Delta=\begin{vmatrix}1 & -1 & 1\\ -1 & 2 & -1\\ 3 & 0 & -2\end{vmatrix}$

$|D|=1\begin{vmatrix}2 & -1\\ 0 & \frac{-1}{3}\end{vmatrix} +1\begin{vmatrix}-1 & -1\\ \frac{1}{2} & \frac{-1}{3}\end{vmatrix} +1\begin{vmatrix}-1 & 2\\ \frac{1}{2} & 0\end{vmatrix}$ √

$|D|=1(\frac{-2}{3})+1(\frac{5}{6})+1(-1)$

$|D|=\frac{5}{6}$ √

$|Dx|=\begin{vmatrix}-3 & -1 & 1\\ -4 & 2 & -1\\ 1 & 0 & \frac{-1}{3}\end{vmatrix}$ √

$|Dx|=-3\begin{vmatrix}2 & -1\\ 0 & -\frac{1}{3}\end{vmatrix} +1\begin{vmatrix}-4 & -1\\ 1 & -\frac{1}{3}\end{vmatrix} +1\begin{vmatrix}-4 & 2\\ 1 & 0\end{vmatrix}$ √

$|Dx|=-3(\frac{-2}{3})+1(\frac{7}{3})+1(-2)$

$|Dx|=\frac{7}{3}$ √

$$x = \frac{|Dx}{|D} = \frac{7}{3} = -\frac{14}{\underrightarrow{5}}$$ (8)

1.3.2 $\begin{vmatrix} 2 & -1 \\ 0 & -\frac{1}{3} \end{vmatrix}$ √

$-\frac{2}{3}$ √ (2)

1.4 $\left.\begin{aligned} 2\ell + 2b = 28............(1) \\ \ell b = 46..................(2) \end{aligned}\right)$ √

In (1) $\ell = 14 - b$

In (2) $(14 - b)b = 46$

$14b - b^2 = 46$

$b^2 - 14b + 46 = 0$ √

$b = \frac{+14 \pm \sqrt{12}}{2}$

$b_2 = 5{,}268$ √

$\ell_2 = 8{,}732$ √ (4)

[20]

QUESTION 2

2.1 $a = \frac{-2 \pm \sqrt{(2)^2 - 4(2)(1)}}{4}$ √

$a = \frac{-2 \pm \sqrt{-4}}{4}$ √

$a = \frac{-2 \pm j2}{4}$ √

$a_1 = \frac{-1}{2} + \frac{j}{2}$ √ $a_2 = -\frac{1}{2} - \frac{j}{2}$ √ (5)

2.2 $\dfrac{20+4j-10j-2j^2}{3-2j} \quad X \quad \dfrac{3+2j}{3+2j}$ √

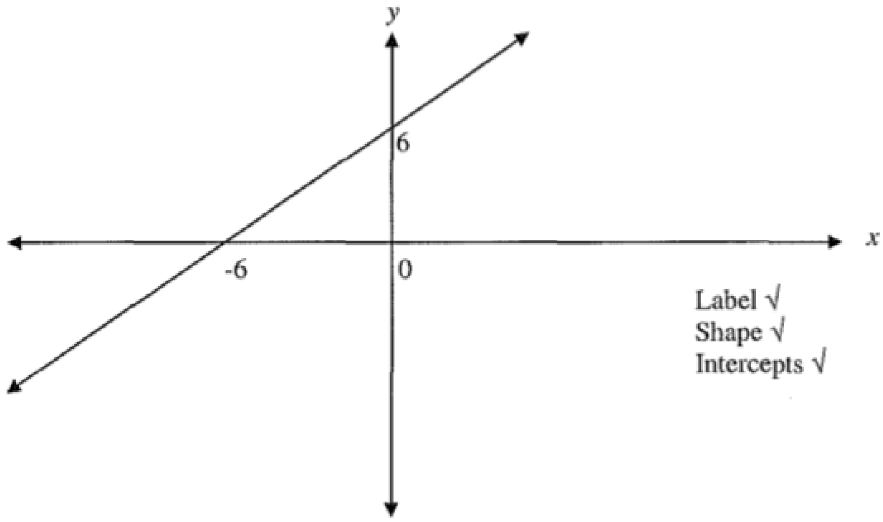
$\dfrac{22-6j}{3-2j} \quad X \quad \dfrac{3+2j}{3+2j}$ √

$\dfrac{66-j18+j44-12j^2}{9-4j^2}$ √

$\dfrac{78+j26}{13}$ √

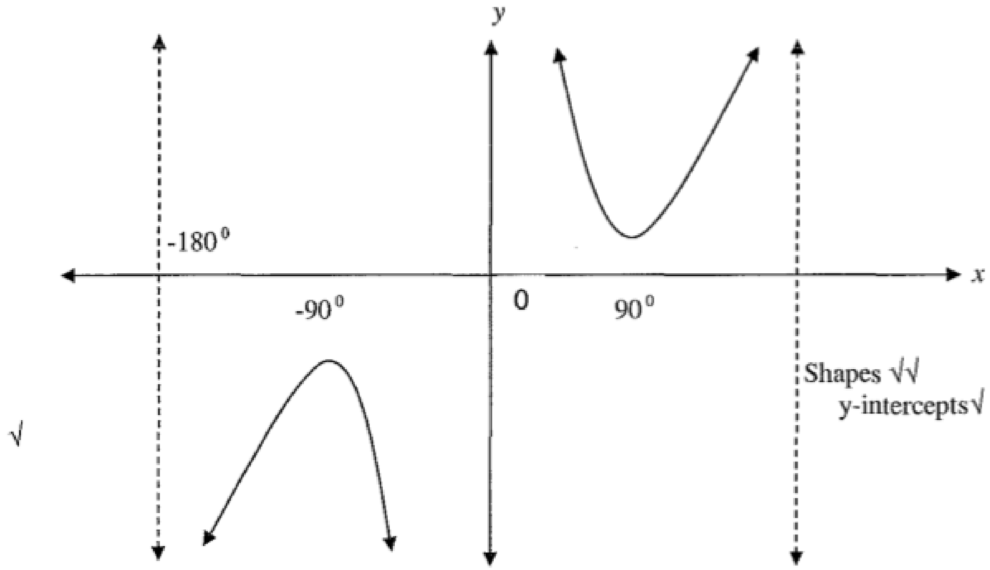
$6+j2$ √ **(5)**

2.3



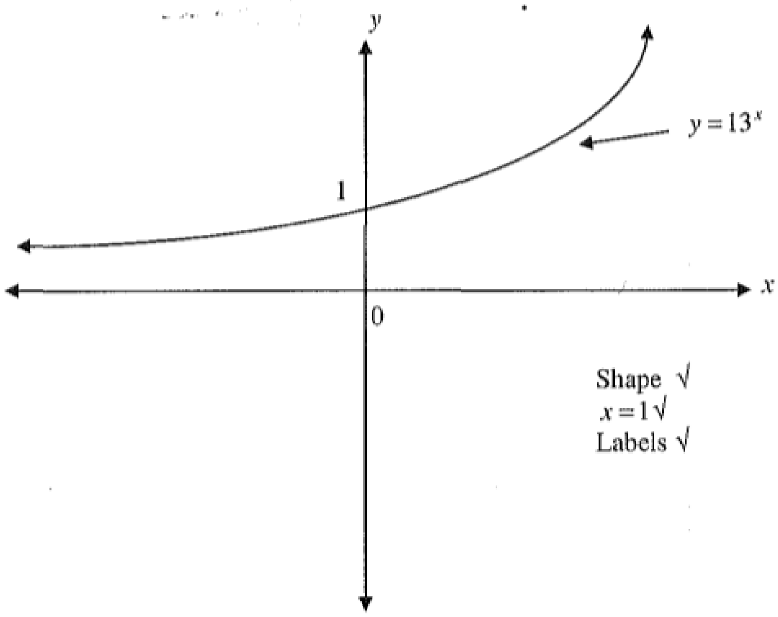
(3)

2.4



(3)

2.5 2.5.1



(3)

2.5.2 Function √ (1)

[20]

QUESTION 3

3.1 $\tan 75^0 = \tan(45^0 + 30^0)$

$= \dfrac{\tan 45^0 + \tan 30^0}{1 - \tan 45^0 . \tan 30^0}$ √

$= \dfrac{\frac{\sqrt{2}}{\sqrt{2}} + \frac{1}{\sqrt{3}}}{1 - \frac{\sqrt{2}}{\sqrt{2}} \bullet \frac{1}{\sqrt{3}}}$ √

$= \dfrac{1 + \frac{1}{\sqrt{3}}}{1 - \frac{1}{\sqrt{3}}} = \dfrac{\sqrt{3} + 1}{\sqrt{3}} \text{ X } \dfrac{\sqrt{3}}{\sqrt{3} - 1}$ √

$= \dfrac{\sqrt{3} + 1}{\sqrt{3} - 1}$ √

(4)

3.2 $= \frac{1-\cos^2\theta}{1+\cos\theta} - 1$ √

$= \frac{(1+\cos\theta)(1-\cos\theta)-1}{1+\cos\theta}$ √

$= 1 - \cos\theta - 1$ √

$= -\cos\theta$ √ (4)

3.3 $\sin 2y = \frac{2\left(\frac{\sin y}{\cos y}\right)}{\sec^2 y}$ √ 2tany $= \frac{2\ \sin\ \ y\ \cos\ \ y}{\sec\ ^2\ \ y}$

$\sin 2y = \frac{\frac{2\sin y}{\cos y}}{\frac{1}{\cos^2 y}}$ √ 2tany $= \frac{2\ \sin\ \ y\ \cos\ \ y}{\cos\ ^2\ \ y}$

$\sin 2y = \frac{2\sin y}{\cos y} \text{X} \frac{\cos^2 y}{1}$ √ 2tany $= 2\left(\frac{smy}{\cos y}\right)$

$\sin 2y = 2\sin y\cos y$ √ ∴ 2tany = 2tany

$\sin 2y = \sin 2y$ (4)

3.4 $\sin(3A+20^0) = \sin[90^0 - (A+10^0)]$ √

$\sin(3A+20^0) = \sin(80^0 - A)$ √

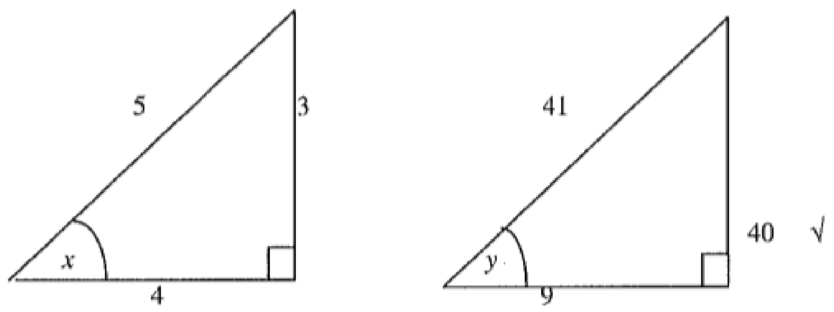
$3A + 20^0 = 80^0 - A$

$4A = 60^0$

$A = 60^0$ √

$A = 15^0$ √ (4)

3.5



$$\sin(x - y) = \sin x \cos y - \sin y . \cos x$$

$$= \frac{3}{5} \cdot \frac{9}{41} - \frac{40}{41} \cdot \frac{4}{5} \quad \surd$$

$$= \frac{27}{205} - \frac{160}{205} \quad \surd$$

$$= \frac{-133}{205} \quad \surd$$

(4)
[20]

QUESTION 4

4.1 $(2x^2)^5 + 5(2x^2)\ (x-3)^4\ \frac{4 \text{ X } 5 (2x^2)^3 (9)}{2} + \frac{3 \text{ X } 4 \text{ X } 5 (2x^2)^2 (-27)}{6}$

$32x^{10} - 240x^8 + 720x^6 - 1080x^4 + ...$ (4)

√ √ √ √

4.2 $\frac{dy}{dx} = \frac{1}{3} . \frac{1}{x} + (-1). - 3x^{-2} - 2\sin x - \frac{7}{5}\cos ecx \cot x$

$= \frac{1}{3x} + 3x^{-2} - 2\sin x - \frac{7}{5}\cos ecx \cot x$

√ √ √ √

(4)

4.3 $U = \sec x \qquad U^1 = \sec x \tan x$ √

$V = \log x \qquad V^1 = \frac{1}{x\ell n10}$ √

$$\frac{dy}{dx} = \frac{\ell ogx \ \sec\ x \tan\ x - \sec\ x \frac{1}{x\ell n10}}{(\ell ogx\ )^2} \ / \ \frac{\sec\ x(\tan\ x \log\ x - \frac{1}{x\ell n10})}{(\ell ogx\ )^2}$$ √√ (4)

4.4 $\frac{dy}{dx} = -6x^2 + 6x + 12$ √

$-6x^2 + 6x + 12 = 0$

$(x-2)(x+1) = 0$ √

$x_1 = 2$ and/en $x_2 = -1$ √

$y_1 = -2(2)^3 + 3(2)^2 + 12(2) - 7$ √ √

$y_1 = 13$

$y_2 = -2(-1)^3 + 3(-1)^2 + 12(-1) - 7$

$y_2 = -14$ √

$(2,13)(-1,-14)$ √

(8)

[20]

QUESTION 5

5.1 $\int \sin x dx$ √

$-\cos x + c$
√ √

(3)

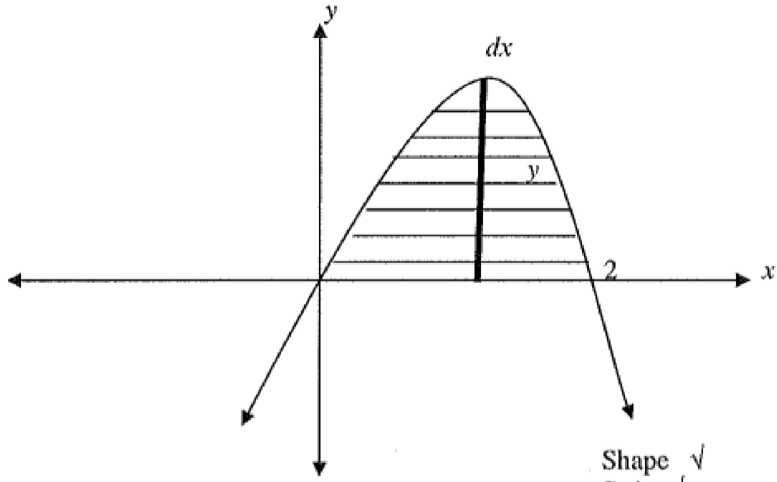
5.2 $= [\sin 4x]_0^{\frac{\Pi}{3}}$

$= \{\sin 4(\frac{\Pi}{3}) - \sin 4(0)$

$= \sin \frac{4\Pi}{3} - \sin 0$

$= -0,8667$ (3)

5.3 5.3.1



Shape √
Strip √
Area shown √ (3)

5.3.2

$$\int_0^2 \left(-x^2+2x\right)dx \quad \surd$$

$$=\left[\frac{-x^3}{3}+x^2\right]_0^2 \quad \surd$$

$$=\left(\frac{-2^3}{3}+(2)^2\right)-(0) \quad \surd$$

$$=1{,}34.u^2 \quad \surd$$

(4)

5.4

$$\frac{\sec\ x}{3}+\frac{2e^{-2}}{3}+\frac{2}{3}x^{\frac{3}{2}}-x^{-4}\ \frac{-2.4^{3x}}{3\ell n4}+px+c$$

√ √ √ √ √ √ √ (7)

[20]

TOTAL: 100

Past Examination Papers

higher education & training

Department: Higher Education and Training
REPUBLIC OF SOUTH AFRICA

APRIL 2014

NATIONAL CERTIFICATE

MATHEMATICS N4

(16030164)

27 March 2014 (Y-Paper)
13:00 – 16:00

Nonprogrammable calculators may be used.

This question paper consists of 5 pages and 1 formula sheet.

DEPARTMENT OF HIGHER EDUCATION AND TRAINING
REPUBLIC OF SOUTH AFRICA
NATIONAL CERTIFICATE
MATHEMATICS N4
TIME: 3 HOURS
MARKS: 100

INSTRUCTIONS AND INFORMATION

1. Answer ALL the questions.
2. Show ALL the calculations and intermediary steps.
3. Simplify where possible.
4. Do ALL graph work in the answer book. Graph paper is NOT supplied.
5. Show ALL values of intercepts with the system of axes and ALL turning point(s) on the graph.
6. Approximate ALL final answers accurately to THREE decimal places.
7. Questions may be answered in any order but DO NOT separate subsections of questions.
8. The attached formula sheet is NOT necessarily a complete list of all formulae.
9. You are NOT compelled to use the formulae on the attached formula sheet.

QUESTION 1:

1.1 Solve for x and y if: (5)

$$x + jy = \frac{5 + j^7}{1 - j}$$

1.2 Given:

$$Z = -3 - j7$$

1.2.1 Convert Z into polar form. θ may only be positive. Show all calculations (3)

1.2.2 Show Z and all the calculated values in 1.2.1 on an Argand diagram. (3)

1.3 Sketch the graph of $xy = 14$ (2)

1.4 1.4.1 Sketch the graph of $x = y^2$ (3)

1.4.2 Is the graph of $x = y^2$ in 1.4.1 a *function* or a *relation*? (1)

1.4.3 Is the graph of $x = y^2$ in 1.4.1 symmetrical about the X-axis? (1)

1.5 Sketch the graph of $x^2 = y^2 = 2$ (2)

[20]

QUESTION 2:

2.1 Solve for x if: (3)

$$\left(\frac{1}{5}\right)^{x-2} = 3.2^{x+1}$$

2.2 Given: (8)

$$3p - 4q = 5$$
$$2p + q - 4 = 0$$

Solve for p and q by use of Cramer's rule.

2.3 The perimeter of a right-angle triangle is 72 cm. The hypotenuse of the triangle is 30 cm. Calculate the lengths of the other two sides. (5)

2.4 Given: (4)

$$V = V_o e^{\frac{1}{T}}$$

Make T the subject of the formula.

[20]

QUESTION 3:

3.1 Show that $\sin \propto = \cos(90° - \propto)$ (3)

3.2 If $\cos A = \frac{3}{5}$ and $B = \frac{12}{13}$ and both angles A and B are acute, calculate WITHOUT the use of a calculator the value of $\cos(A + B)$. (4)

3.3 Simplify as far as possible: (4)

$$\frac{1 + cot^2 x}{\cot x \cos ecx}$$

3.4 Solve for x if: (5)

$$5\,sin^2\,x - 5\cos x = -3; \quad 0° \leq x \leq 360°$$

3.5 Prove that: (4)

$$\frac{2\sin y + 1}{\sin 2y + \cos y}$$

[20]

QUESTION 4:

4.1 Differentiate the following: (6)

$$y = \frac{\sin 2x + 2}{2\cos x} - 3.7^x - 8e^{7x} + x^{\frac{1}{3}} - p$$

4.2 Differentiate from first principles if: (5)

$$y = 2x^2 + 3x$$

4.3 Differentiate by using a function of a function rule (chain rule) if: (4)

$$y = 6e^{3x}$$

4.4 Given: $y = x^3 + 12x^2 - 36x + 11$ (5)

[20]

QUESTION 5:

5.1 Integrate the following: (7)

$$\int \left(-4x^{-2} - \frac{4}{x} - \frac{e^x}{4} - 10^{4x} + \frac{1}{\sqrt{x}} + 3\cos ecx \cot x\right) dx$$

5.2 Simplify: (3)

$$\int \left(\frac{x^3 - 27}{x - 3}\right) dx$$

5.3 Evaluate: (3)

$$\int_1^2 (5x^4 + 4x^3) dx$$

5.4 5.4.1 Sketch the graph of $y = 5\sin x$, $x = 0$ and $x = \frac{n}{2}$ and indicate the enclosed area. Also, indicate the representative strip used to calculate the enclosed area. (3)

5.4.2 Calculate the value of the indicated area in 5.4.1 by using integration. (4)

[20]

TOTAL: 100

MATHEMATIC N4

FORMULA SHEET

NEW SYLLABUS

$a^x = b \Leftrightarrow \log a^x = \log b$

$\ell n\, x = \log_e x$

$(r \underline{|\theta})^n = r^n \underline{|n\theta}$ $\quad a + bj = c + dj \Leftrightarrow a = c$ and $b = d$

$\sin(a \pm b) = \sin a \cos b \pm \sin b \cos a$
$\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$

$\sin^2 x + \cos^2 x = 1$
$1 + \cot^2 x = \operatorname{cosec}^2 x$
$1 + \tan^2 x = \sec^2 x$

$$\tan(a \pm b) = \frac{\tan a \pm \tan b}{1 \mp \tan a \tan b}$$

y	$\frac{dy}{dx}$
ax^n	nax^{n-1}
ka^x	$ka^x \ell na$
$k\ell nx$	$\frac{k}{x}$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\sec x$	$\sec x \tan x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$

$y = u(x) \cdot v(x)$

$$\Rightarrow \frac{dy}{dx} = u(x)v'(x) + u'(x)\,v(x)$$

$$y = \frac{u(x)}{v(x)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{v(x)u'(x) - u(x)v'(x)}{[v(x)]^2}$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\int ax^n\,dx = \frac{ax^{n+1}}{n+1} + C$$

$$\int \sin x\,dx = -\cos x + c$$

$$\int \frac{a}{x}\,dx = a\ell n\,x + c$$

$$\int \cos x\,dx = \sin x + c$$

$$\int ka^x dx = \frac{ka^x}{\ell na} + c$$

$$\int \tan x\,dx = \ell n \sec x + c$$

$$\int \sec x\,dx = \ell n(\sec x + \tan x) + c$$

$$A_{ox} = \int_a^b y\,dx$$

Marking Guidelines

higher education & training

Department:
Higher Education and Training
REPUBLIC OF SOUTH AFRICA

APRIL 2014

NATIONAL CERTIFICATE

MATHEMATICS N4

(16030164)

27 March 2014 (Y-Paper)
13:00 – 16:00

QUESTION 1

1.1 $x + yj = \frac{5-j}{1-j}$

$(x+yj)(1-j) = 5-j$ √

$x - xj + yj - j^2 y = 5 - j$

$\left.\begin{array}{l} x + y = 5 \ldots\ldots\ldots\ldots\ldots(1) \\ -x + y = -1 \ldots\ldots\ldots\ldots(2) \end{array}\right)$ √

$2y = 4$

$y = 2$ √

In (1) $x + 2 = 5$ √

$x = 3$ √ (5)

1.2

1.2.1 $r = \sqrt{(-3)^2 + (-7)^2}$

$r = \sqrt{58}$

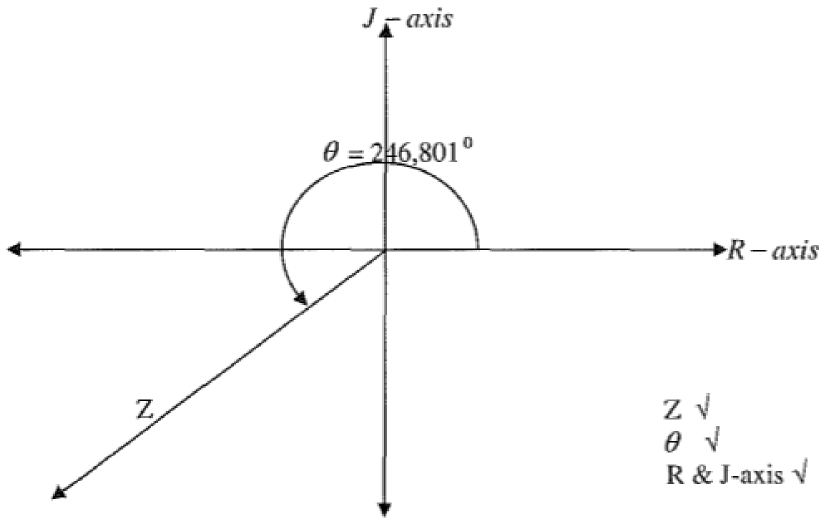
$r = 7{,}616$ √

$\theta = 180^0 + \tan^{-1}\frac{7}{3}$

$\theta = 66{,}80P + 180^0 = 246{,}801^0$ √

$Z = 7{,}616\ \underline{|246{,}801^0}$ √ (3)

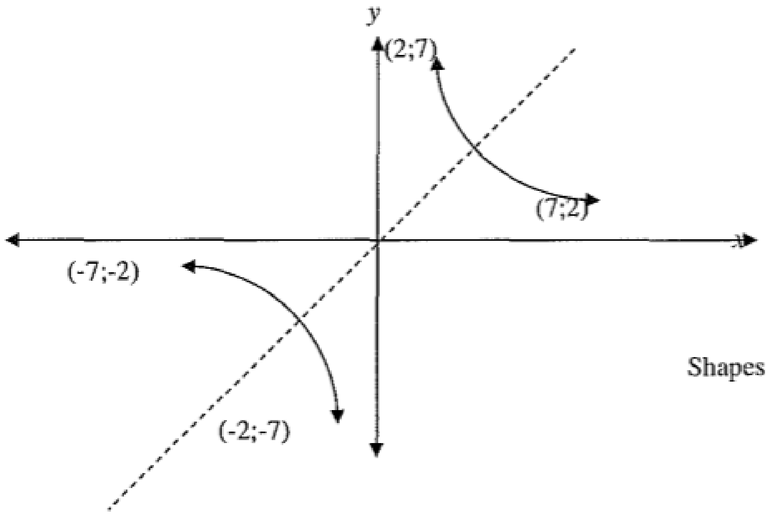
1.2.3



Z √
θ √
R & J-axis √

(3)

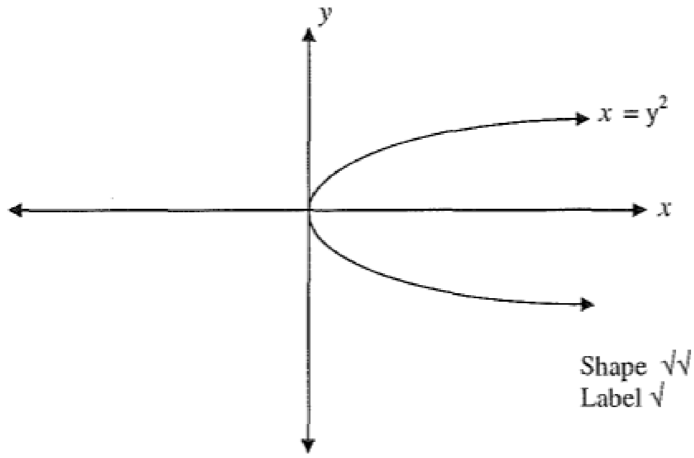
1.3



Shapes √√

(2)

1.4 1.4.1

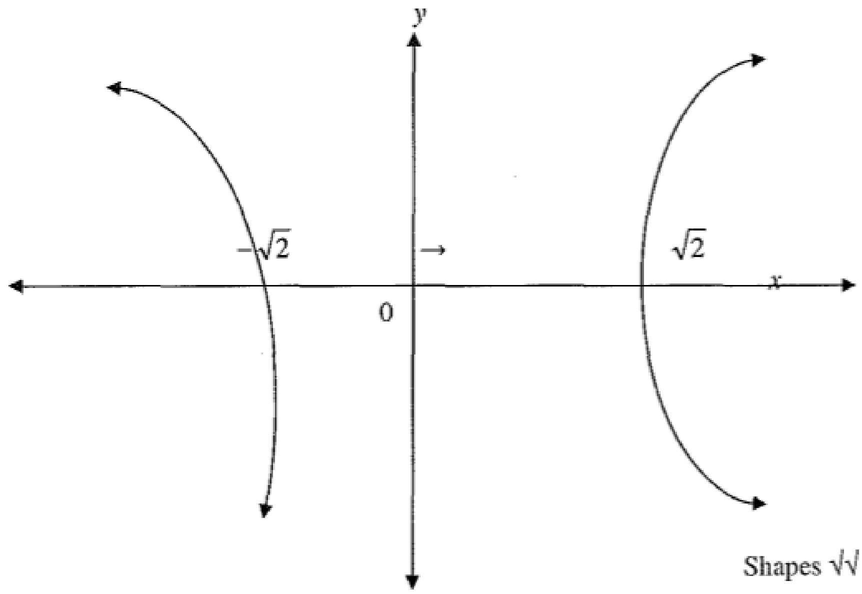


(3)

1.4.2 Function √ (1)

1.4.3 Yes √ (1)

1.5



(2)

[20]

QUESTION 2

2.1 $(x-2)\ell n\left(\frac{1}{5}\right)=\ell n3+(x+1)\ell n2$ √

$-1{,}609x+3{,}219=1{,}099+0{,}693x+0{,}693$ √

$-2{,}302x=-1{,}427$

$x=0{,}62$ √ (3)

2.2 $|D|=\begin{vmatrix} 3 & -4 \\ 2 & 1 \end{vmatrix}$ √

$|D|=11$ √

$|Dp|=\begin{vmatrix} 5 & -4 \\ 3 & 1 \end{vmatrix}$ √

$=21$ √

$|Dq|=\begin{vmatrix} 3 & 5 \\ 2 & 4 \end{vmatrix}$ √

$=2$ √

$p=\frac{|Dp|}{|D|}=\frac{21}{11}/1{,}91$ √

$q=\frac{|Dq|}{D}=\frac{2}{11}/0{,}182$ √ (8)

2.3 $\left.\begin{array}{l} x^2+y^2=900\text{.................(1)} \\ x+y=42\text{.....................(2)} \end{array}\right\}$ √

In (2) $x=42-y$

In (1) $(42-y)^2+y^2=900$

$y^2-42y+432=0$ √

$(y-18)\ (y-24)=0$ √

$y_1=18$ and $y_2=24$ √

In (2) $x_1+18=42$ and $x_2+24=42$

$x_1=24$ and $x_2=18$ √ (5)

2.4 $\frac{V}{V_0}=e^{-\frac{t}{T}}$ √

$\ell n e^{-\frac{t}{T}}=\ell n\left(\frac{V}{V_0}\right)$ √

$-\frac{t}{T}=\ell n\left(\frac{V}{V_0}\right)$ √

$T=\frac{-t}{\ell n\left(\frac{V}{V_0}\right)}$ √

(4)
[20]

QUESTION 3

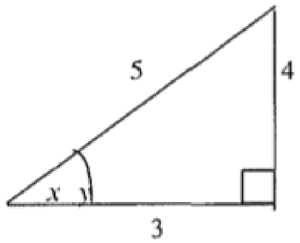
3.1 $\cos(90^0-\alpha)=\sin\alpha$

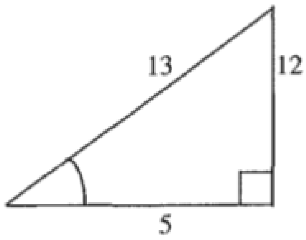
$\cos 90^0\cos\alpha+\sin 90^0.\sin\alpha=\sin x$ √

$0.\ \cos\alpha+1.\sin\alpha=\sin\alpha$ √

$\sin\alpha=\sin\alpha$ √ (3)

3.2





$\cos(x+y) = \cos x \cos y - \sin x \sin y$

$= \frac{3}{5}.\frac{5}{13} - \frac{4}{5}.\frac{12}{13}$ √√

$= \frac{15}{65} - \frac{48}{65}$ √

$= \frac{-33}{65}$ √ (4)

3.3

$= \frac{\cos ec^2 x}{\cot x \cos ecx}$ √

$= \frac{\cos ecx}{\cot x}$

$= \frac{\frac{1}{\sin x}}{\frac{\cos x}{\sin x}}$ √

$= \frac{1}{\sin x} \times \frac{\sin x}{\cos x}$ √

$= \frac{1}{\cos x}$

$= \sec x$ √ (4)

3.4 $5(1-\cos^2 x)-5\cos x=-3$ √

$5\cos^2 y+5\cos y \quad 8=0$

$$\cos y=\frac{5+\sqrt{(5)^2 \quad 4(5)(\quad 8)}}{10} \quad √$$

$\cos y=0{,}860$ √ and $\cos y=1{,}860$

$y_1=30{,}683^0$ √ $\quad$ = Invalid

$y_2=329{,}317^0$ √ (5)

3.5 $\dfrac{2\sin y+1}{2\sin y\cos y+\cos y}=\sec y$ √

$\dfrac{2\sin y+1}{\cos y(2\sin y+1)}=\sec y$ √

$\dfrac{1}{\cos y}=\sec y$ √

$\therefore \sec y=\sec y$ √ (4)

[20]

QUESTION 4

4.1 $y=\sin x+\sec x-3{,}7^x-p-8e^{7x}+x^{\frac{1}{3}}$

$$\frac{dy}{dx}=\cos x+\sec x\tan x-3{,}7^x\ell n7-0-56e^{7x}+\frac{1}{3}x^{-\frac{2}{3}}$$

√ √ √ √ √ √ (6)

4.2 $f(x) = 2x^2 + 3x$

$f(x+h) = 2(x+h)^2 + 3(x+h)$

$= 2(x^2 + 2xh + h^2) + 3x + 3h$

$= 2x^2 + 4xh + 2h^2 + 3x + 3h$

$f(x+h) - f(x) = 2x^2 + 4xh + 2h^2 + 3x + 3h \quad 2x^2 - 3x$ √

$= 4xh + 2h^2 + 3h$ √

$\dfrac{f(x+h) - f(x)}{h} = \dfrac{4xh + 2h^2 + 3h}{h}$ √

$= 4x + 2h + 3$ √

$\ell im\left(\dfrac{f(x+h-f(x))}{h}\right) = 4x + 0 + 3$

$h \rightarrow 0 \quad = 4x + 3$ √ (5)

4.3 $\ell et \quad u = 3x$

$y = 6e^u \qquad u = 3x$

$\dfrac{dy}{du} = 6e^u$ √ $\dfrac{du}{dx} = 3x$ √ OR $\dfrac{du}{dx} = 6e^{3x}$

$= 3.6e^{3x}$

$= 18e^{3x}$

$\dfrac{dy}{dx} = \dfrac{du}{dx} \text{ X } \dfrac{dy}{du}$

$\dfrac{dy}{dx} = 3x \text{ X } 6e^u$ √

$\dfrac{dy}{dx} = 18xe^u$

$= 18xe^{3x}$ √ (4)

4.4 $\frac{dy}{dx} = 3x^2 + 24x - 36$ √

$\frac{d^2y}{dx^2} = 6x + 24$ √

$\frac{d^2y}{dx^2} = 0$ √

$6x + 24 = 0$

$x = -\frac{24}{6}$ or $\frac{-12}{3}$ √

$x = -4$

$y = (-4)^3 + 12(-4)^2 - 36(-4) + 11$

$= -64 + 192 + 144 + 11$

$= 283$

P.O.I = (–4; 283) (5)

[20]

QUESTION 5

5.1

$\frac{4}{x} - 4\ell n x - \frac{e^x}{4} - \frac{10^{4x}}{4\ell n 10} + 2\sqrt{x} - 3\cos ec x + c$

√ √ √ √ √ √ (7)

5.2 $\frac{x^3}{3} + \frac{3x^2}{2} + 9x + c$

√ √ √ (3)

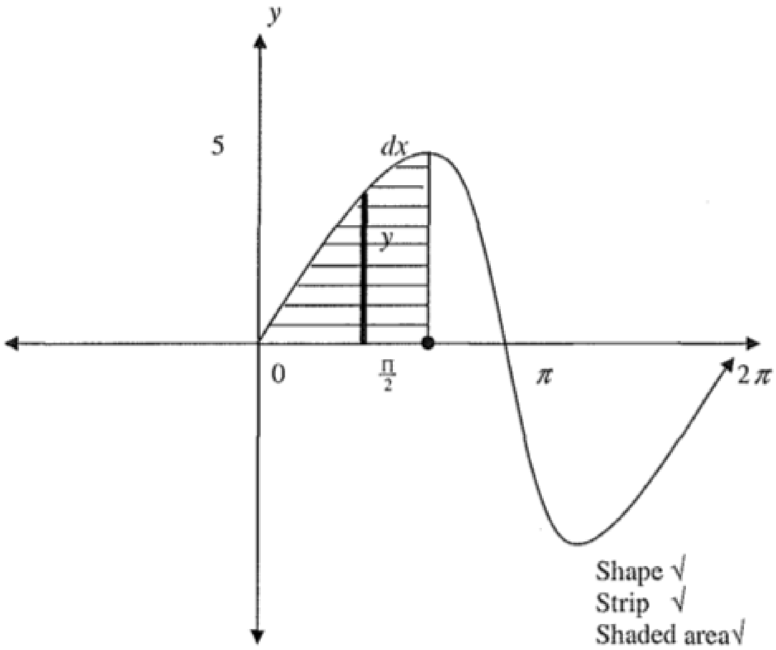
5.3 $= \left[x^5 + x^4\right]_1^2$ √

$= \left[(2^5 + 2^4) - (1^5 + 1^4)\right]$ √

$= 48 - 2$

$= 46$ √ (3)

5.4 5.4.1



(3)

5.4.2

$$AoX = \int_0^{\frac{\Pi}{2}} (5\sin x)dx \quad \surd$$

$$= [-5\cos x]_0^{\frac{\Pi}{2}} \quad \surd$$

$$= [(-5\cos\tfrac{\Pi}{2} - (-5\cos 0)] \quad \surd$$

$$= 0 + 5$$

$$= 5u^2 \quad \surd$$

(4)
[20]

TOTAL: 100

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